Persistency of 2D perturbations of one-dimensional solutions for a Cahn-Hilliard flow model under high shear

Franck Boyer, Pierre Fabrie

Persistency of 2D perturbations of one-dimensional solutions for a Cahn-Hilliard flow model under high shear

Franck Boyer[1], **Pierre Fabrie**[2],

Abstract - We consider a diphasic system in a high shear regime when separation of phases occurs. One can observe that the two phases organize themselves into numerous bands, parallel to the flow direction. Mathematically, these are solutions of a certain system of equations depending only upon the transversal variable (1D solutions). We study the stability of these 1D solutions with respect to 2D pertubations. The mathematical model used in our analysis is a coupling between a Cahn-Hilliard equation and the Navier-Stokes equations in two dimensions. We show that a small 2D perturbation of a given 1D solution persists for significant times. We give the precise size of such a perturbation and its time of persistence. Moreover, we obtain an asymptotic expansion of the solution in the considered cases. Note that, for a mathematical model to be realistic one has to take into account the fact that in experiments the high shear regime is obtained in elongated domains (a very thin Couette cell for example). Therefore, we perform the mathematical analysis of this problem in a stretched domain.

Key Words - Cahn-Hilliard equation, Navier-Stokes equation, Diphasic flows, Shear flow, Spinodal Decomposition

AMS Subject Classification - 35B40, 76D05, 76D45, 76T99

1 Introduction

• **Cahn-Hilliard models for diphasic flows:**

This paper is concerned with a qualitative study of the behavior of a diphasic system in a high shear regime. Among the numerous existing models for the study of incompressible mixture flows, a new approach was presented in [8] following, by example, the previous works by Cahn and Hilliard [9], Doi [14] and Onuki [21]. Its physical relevance has been shown through various numerical simulations in [8]. Several theoretical results were obtained in [6, 7]. This model takes into account a diffuse interface between the two phases, that is to say that the interface is supposed to have a small but non-zero thickness. We refer to [2], for a review concerning this kind of approaches and to [17] for a study of a similar model.

In the interfacial zone, the various quantities describing the mixture are supposed to vary very quickly but continuously. This is achieved by introducing a free energy of the mixture which not only depends upon the composition of the blend but also on its gradient (Van der Waals, Cahn and Hilliard [9]). In a dimensionless form, this free energy reads

$$\mathcal{F}(\tilde{\varphi}) = \int_\Omega \frac{\alpha^2}{2}|\nabla\tilde{\varphi}|^2 + F(\tilde{\varphi}),$$

where $\tilde{\varphi}$ characterizes the composition of the mixture (the volumic part of one of the constituent), α is the dimensionless interfacial thickness, and F represents the Cahn-Hilliard potential with double-well shape as shown in Figure 1.1.

The Cahn-Hilliard theory is quite classical in the study of phase transitions. The stationary theory leads to numerous problems of minimization of the Cahn-Hilliard free energy under a given mass $\int_\Omega \tilde{\varphi} = M$ constraint (see for example [3], [10], [24]). The non-stationary Cahn-Hilliard theory leads to a parabolic fourth order equation which may eventually degenerate (see [6], [7], [15], [13]). In fact this equation can be generalized to different situations in material sciences: viscous Cahn-Hilliard equation [20], deformable medium theory (Cahn-Hilliard-Gurtin models) [18], [19].

Using the Cahn-Hilliard theory to describe the conservation laws (of mass and momentum) in each phase of a binary flow, a mathematical model for a diphasic flow has been derived in [8]. The two phases are supposed to have the same

[1]LATP, CMI, UMR 6632, Université Aix-Marseille 1, 39 rue Joliot-Curie, 13453 Marseille Cedex 13 FRANCE,
e-mail: `fboyer@cmi.univ-mrs.fr`
[2]Mathématiques Appliquées de Bordeaux, UMR 5466, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex FRANCE,
e-mail: `fabrie@math.u-bordeaux.fr`

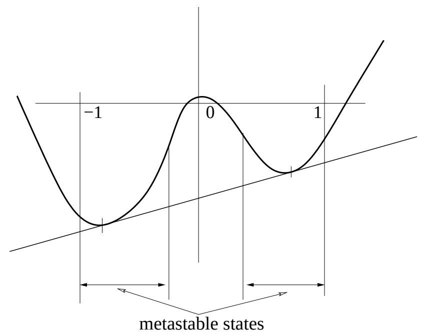

Figure 1.1: Cahn-Hilliard potential F

densities and the system of equations consists of a coupling between an incompressible Navier-Stokes equation for the velocity field $\tilde{V}$ and a Cahn-Hilliard equation for the order parameter $\tilde{\varphi}$. These equations are in a general form

$$\partial_t\tilde{\varphi} + \tilde{V}.\nabla\tilde{\varphi} - \tfrac{1}{\mathcal{P}e}\,\mathrm{div}\,(B(\tilde{\varphi})\nabla\tilde{\mu}) = 0, \tag{1.1}$$

$$\tilde{\mu} = -\alpha^2\Delta\tilde{\varphi} + F'(\tilde{\varphi}), \tag{1.2}$$

$$\partial_t\tilde{V} + \tilde{V}.\nabla\tilde{V} - \tfrac{2}{\mathcal{R}e}\,\mathrm{div}\,(\eta(\tilde{\varphi})D(\tilde{V})) + \nabla\tilde{p} = \mathcal{K}\tilde{\mu}\nabla\tilde{\varphi}, \tag{1.3}$$

$$\mathrm{div}\,(\tilde{V}) = 0, \tag{1.4}$$

where $\mathcal{P}e$ is the Peclet number, $\mathcal{R}e$ the Reynolds number and $\mathcal{K}$ a capillarity coefficient. Finally $B(\tilde{\varphi})$ and $\eta(\tilde{\varphi})$ are respectively the mobility and the viscosity. As usual, $D(\tilde{V})$ denotes the rate of deformation tensor $(\nabla\tilde{V} + \nabla\tilde{V}^t)/2$.

- **Shear flows:**

In this paper, we are particularly interested in the behavior of such a system in a high shear flow. Let us consider, for instance, the two following physical situations: a Couette flow in a rheometer and a plane shear flow in a plate-plate cell (Figure 1.2). These two systems are used to investigate the rheological properties of a mixture. In each case, the system is composed of two parts. One is static (the stator) and one is in rotation with constant angular velocity (the rotor).

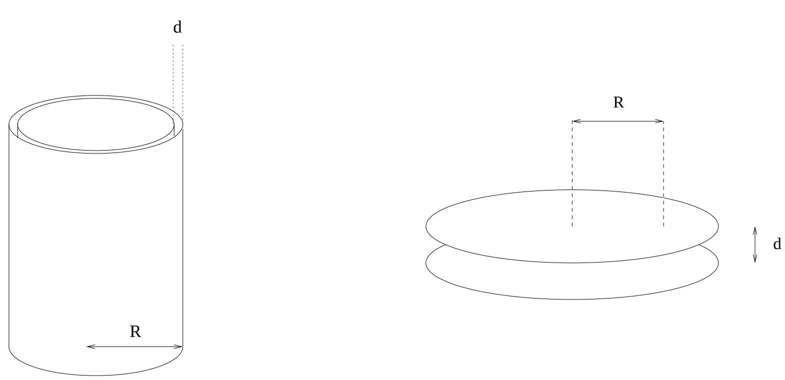

Figure 1.2: Couette flow in a rheometer and Shear flow in a plate-plate cell

In the two cases, the width of the cell d is the characteristic size of the domain. Hence, in dimensionless variables we take $d = 1$. The constant R is the radius of the cell in the rheometer case, or the distance between the center and the observation point in the plate-plate case. In both cases, R is large compared to d so that in dimensionless variables we write $R = \frac{1}{\varepsilon}$, with ε small. The time scale is chosen so that the angular velocity of the system is $\omega = 1$. The linear velocity on the rotor is then given in dimensionless variables by

$$\tilde{V} = \omega . R = \frac{1}{\varepsilon},$$

and the shear rate by

$$\dot{\gamma} = \frac{\tilde{V}}{d} = \frac{1}{\varepsilon}.$$

The high shear regime corresponds to the case ε small.

The interaction between thermodynamics (formation and evolution of patterns in a binary mixture) and hydrodynamics (high shear flow) has numerous consequences from a physical point of view. One of them, observed in the physical experiments [4, 5, 12, 16] but also in numerical simulations [8], is the following. In a mixture (under its critical point) submitted to a shear flow, for instance in a system just like the ones presented in figure 1.2, two phenomena may occur: the progressive separation of the two phases (spinodal decomposition) and the organization of the created patterns into parallel layers to the flow direction.

It has been shown, experimentally and numerically [16, 8], that as the shear rate is increased, the qualitative evolution of the system may radically change. In particular, at high shear, the critical point seems to be changed, *i.e.* at a given temperature, the shear flow stabilizes a mixture which is unstable at rest.

• **Mathematical analysis:**

(proportional to the volumic part of one phase in the mixture) Our goal is to prove that the system (1.1)-(1.4) is able to describe some of these properties. We are particularly interested in the stability properties of the one-dimensional solutions of the problem, more precisely of the solutions depending only upon the variable z transversal to the flow direction. These solutions correspond exactly to the parallel layers mentioned above.

In section 2 we discuss the stability of these solutions with respect to 2D perturbations. We describe the behavior of the solution, when ε is small enough, for a 2D initial data which is a small perturbation of an 1D initial data. We show that a perturbation of order ε persists on significant time intervals and we validate an asymptotic expansion in ε of the solution. In fact, we establish three different results depending on the particular nature of the 1D solution considered. A constant but non-metastable solution is found to be more stable than a general 1D solution. Of course, a constant and metastable solution is found to be much more stable than the other ones.

From now on, we deal with the general equations (1.1)-(1.4) previously introduced, in the particular case of the high shear regime (see Figure 1.2) and we make the following assumptions:

- The Cahn-Hilliard potential is a polynom, of the form

$$F(\tilde{\varphi}) = \frac{\tilde{\varphi}^4}{4} - \frac{\tilde{\varphi}^2}{2}. \tag{1.5}$$

  This assumption is classical in the Cahn-Hilliard theory (see for example [11, 18, 22]). When the temperature of the system is not too far from the critical point, such a polynomial free energy is a good approximation of a more relevant logarithmic potential.

- The inertial terms are neglected in the Navier-Stokes equation. This assumption corresponds to the case of a laminar regime.

- The mobility and the viscosity are independent of the order parameter $\tilde{\varphi}$ and are set to 1.

Moreover, without loss of generality, we take the parameters $\mathcal{P}e$, $\mathcal{R}e$ and $\mathcal{K}$ equal to 1.

In the remaining of this paper, we model the two physical situations discibed in Figure 1.2 by considering an anisotropic 2D domain given by

$$\Omega_\varepsilon = \left]0, \frac{1}{\varepsilon}\right[\times]0, 1[,$$

which stands for a 2D representation of the two systems in which the gap between the two cylinders (or the two plates) is small compared to the mean radius of the cell. This allows us to neglect the effects of the curvature of the domain. We denote by x the coordinate in the flow direction, and by z the coordinate orthogonal to the flow direction.

We will see further that the key-point of this analysis is to cope with the strong anisotropy of the thin domain Ω_ε that we consider. To achieve this, we will introduce a change of variables to perform the analysis in an ε- independent domain and we will use anisotropic Sobolev inequalities to take into account the anisotropy in the Sobolev inequalities. (see [23]).

The unknowns of the problem in the physical domain Ω_ε are denoted respectively by $\tilde{\varphi}_\varepsilon, \tilde{\mu}_\varepsilon$ and $\tilde{V}_\varepsilon$. We assume that $\tilde{\varphi}_\varepsilon$, $\tilde{\mu}_\varepsilon$ and $\tilde{V}_\varepsilon$ are periodic in the x-direction and satisfy the following boundary conditions:

$$\frac{\partial \tilde{\varphi}_\varepsilon}{\partial \boldsymbol{\nu}} = \frac{\partial \tilde{\mu}_\varepsilon}{\partial \boldsymbol{\nu}} = 0, \text{ on } \{z=0\} \cup \{z=1\},$$
$$\tilde{V}_\varepsilon = 0, \text{ for } z=0, \text{ and } \tilde{V}_\varepsilon = \frac{1}{\varepsilon} e_x, \text{ for } z=1.$$

The above boundary conditions on $\tilde{V}_\varepsilon$ lead to a unique stationary solution of the Stokes equation in Ω_ε given by $\dfrac{z}{\varepsilon}e_x$. We are interested in the perturbations of such stationary states of the velocity , so we set

$$\tilde{V}_\varepsilon = \frac{z}{\varepsilon}e_x + \tilde{U}_\varepsilon,$$

e_x being the unitary vector in the x-direction, and $\tilde{U}_\varepsilon$ being small compared to $\frac{1}{\varepsilon}$. After this change of function, the system (1.1)-(1.4) in the domain Ω_ε becomes:

$$\partial_t \tilde{\varphi}_\varepsilon + \frac{z}{\varepsilon}\partial_x \tilde{\varphi}_\varepsilon + \tilde{U}_\varepsilon . \nabla \tilde{\varphi}_\varepsilon - \Delta \tilde{\mu}_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \tag{1.6}$$
$$\tilde{\mu}_\varepsilon = -\alpha^2 \Delta \tilde{\varphi}_\varepsilon + F'(\tilde{\varphi}_\varepsilon) \text{ in } \Omega_\varepsilon, \tag{1.7}$$
$$\partial_t \tilde{U}_\varepsilon - \Delta \tilde{U}_\varepsilon + \nabla \tilde{p}_\varepsilon = -\alpha^2 \Delta \tilde{\varphi}_\varepsilon \nabla \tilde{\varphi}_\varepsilon \text{ in } \Omega_\varepsilon, \tag{1.8}$$
$$\text{div } (\tilde{U}_\varepsilon) = 0, \tag{1.9}$$

with periodicity and homogeneous boundary conditions on $\tilde{\varphi}_\varepsilon$, $\tilde{\mu}_\varepsilon$ and $\tilde{U}_\varepsilon$.

Remark: In order to simplify the computations in the sequel of the article, we write the capillary forces term under the form $-\alpha^2 \Delta \tilde{\varphi}_\varepsilon \nabla \tilde{\varphi}_\varepsilon$ instead of $\tilde{\mu}_\varepsilon \nabla \tilde{\varphi}_\varepsilon$. The two forms are equivalent because the term $F'(\tilde{\varphi}_\varepsilon)\nabla \tilde{\varphi}_\varepsilon$ may be considered as a part of the pressure gradient for this incompressible problem.

We denote the components of the velocity field by

$$\tilde{U}_\varepsilon = \begin{pmatrix} \tilde{u}_\varepsilon \\ \tilde{v}_\varepsilon \end{pmatrix}.$$

This article is structured in the following way:

- In a first part, we begin with the description of the 1D solutions of the problem (which are the main object of this work). Then, after having introduced some notations, we state the results contained in this article.

- The second part is concerned with some preliminary results that we systematically use in the remaining of the paper.

- Finally, the last three parts deal with the fundamental energy estimates that we need, and with the proofs of the announced results.

2 Statement of the results

- **1D solutions of the system:**

As mentioned above, this work deals with the 1D solutions of problem (1.6)-(1.9), that is to say with the solutions which only depend on the variable z (transversal to the flow direction). Let $(\tilde{\varphi}_0(t,z), \tilde{U}_0(t,z))$ be such a solution, with

$$\tilde{U}_0(t,z) = \begin{pmatrix} \tilde{u}_0(t,z) \\ \tilde{v}_0(t,z) \end{pmatrix}.$$

Then, condition (1.9) on $\tilde{U}_0$ and the fact that $\partial_x \tilde{u}_0 = 0$ imply that

$$\partial_z \tilde{v}_0 = 0.$$

Hence, since $\tilde{U}_0 = 0$ on the boundaries $\{z=0\} \cup \{z=1\}$, it follows that

$$\tilde{v}_0 = 0.$$

The field $\tilde{U}_0$ is directed along the x-direction and $\partial_x \tilde{\varphi}_0 = 0$, so that the transport term disappears in (1.6), which becomes

$$\partial_t \tilde{\varphi}_0 - \partial_z^2 \tilde{\mu}_0 = 0, \tag{2.1}$$
$$\tilde{\mu}_0 = -\alpha^2 \partial_z^2 \tilde{\varphi}_0 + F'(\tilde{\varphi}_0). \tag{2.2}$$

Moreover, equation (1.8) reads in these conditions

$$\partial_t \tilde{u}_0 - \Delta \tilde{u}_0 + \partial_x \tilde{p}_0 = 0,$$
$$\partial_z \tilde{p}_0 = -\alpha^2 (\partial_z^2 \tilde{\varphi}_0)(\partial_z \tilde{\varphi}_0).$$

The pressure and velocity fields given by

$$\tilde{p}_0 = -\frac{\alpha^2}{2} |\partial_z \tilde{\varphi}_0|^2, \tag{2.3}$$

and

$$\partial_t \tilde{u}_0 - \Delta \tilde{u}_0 = 0,$$

are clearly the unique solution of this problem and then, $\tilde{U}_0$ has the following form:

$$\tilde{U}_0 = \begin{pmatrix} \tilde{u}_0 \\ 0 \end{pmatrix}. \tag{2.4}$$

Then, the one-dimensional solutions of system (1.6)-(1.9) are obtained through two uncoupled equations: a classical 1D Cahn-Hilliard equation for $\tilde{\varphi}_0$ and a heat equation for $\tilde{u}_0$. Let us note that these solutions do not depend on ε, that is to say on the shear rate the system is submitted to.

From now on, we are interested in the behavior of system (1.6)-(1.9) for 2D initial data which are close enough to 1D data, in high shear conditions, that is to say for small ε.

- **Linearized equation:**

For $\varepsilon > 0$ given, let $\tilde{\varphi}_1$ (which depends on ε in an implicit way) be the solution of the Cahn-Hilliard equation (1.6)-(1.7) with $\tilde{U}_\varepsilon = \tilde{U}_0$, linearized around the 1D reference solution $\tilde{\varphi}_\varepsilon = \tilde{\varphi}_0$. This linearized equation reads

$$\partial_t \tilde{\varphi}_1 + \frac{z}{\varepsilon} \partial_x \tilde{\varphi}_1 + \tilde{u}_0(t,z) \partial_x \tilde{\varphi}_1 - \Delta \tilde{\mu}_1 = 0, \tag{2.5}$$
$$\tilde{\mu}_1 = -\alpha^2 \Delta \tilde{\varphi}_1 + F''(\tilde{\varphi}_0(t,z)) \tilde{\varphi}_1. \tag{2.6}$$

We will see later that this linear equation define a unique function $\tilde{\varphi}_1$. This function is expected to be a good description of the evolution of the principal part of the perturbation along the time. The following results will precise this statement.

- **Domain invariant norm:**

Since we want to understand the influence of the shear rate $\frac{1}{\varepsilon}$ on the 1D solutions of the problem, we have to state the results using a norm on $L^2(\Omega_\varepsilon)$ for which functions independent of x have norms independent of ε.

Definition 2.1 (Domain invariant norm)
For any function $\tilde{f}_\varepsilon \in L^2(\Omega_\varepsilon)$, let us define the following norm:

$$|\tilde{f}_\varepsilon|_{2,\varepsilon} = \sqrt{\varepsilon}\|\tilde{f}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

In particular, the 1D solutions $\tilde{\varphi}_0$ and $\tilde{u}_0$ have, with this choice, a norm in $L^2(\Omega_\varepsilon)$ which is independent of ε. This fact makes the following results physically meaningful.

We can now state the results obtained in this paper, in each of the three cases under study.

- **General case:**

If we do not make any particular assumption on φ_0 and U_0 (in particular they may depend on time), then we prove the persistency of small perturbations of size ε of this solution along times of order $O(1)$. The main difficulties arise from the nonlinear terms: the Cahn-Hilliard term $F'(\varphi)$, but also the capillary forces term in the Navier-Stokes equation, which are the main cause of the increase of small perturbations of one-dimensional solutions. The precise result is the following.

Theorem 2.1
Let $\tilde{\varphi}_0$ be a 1D solution of the Cahn-Hilliard equation (2.1)-(2.2)*, for a given initial data φ_0^0. We consider a family of 2D initial data of the form*

$$\tilde{\varphi}_\varepsilon(0) = \varphi_0^0 + \varepsilon\tilde{\varphi}_1^0,$$

$$\tilde{U}_\varepsilon(0) = u_0^0(z)e_x + \tilde{W}_\varepsilon^0,$$

for system (1.6)-(1.9) *in the domain Ω_ε. We suppose that $m(\tilde{\varphi}_1^0) = 0$ and that there exists a constant K_0 (depending only on φ_0^0) such that*

$$|\tilde{\varphi}_1^0|_{2,\varepsilon} + \varepsilon^{\frac{1}{2}}|\nabla\tilde{\varphi}_1^0|_{2,\varepsilon} \leq K_0, \tag{2.7}$$

$$|\tilde{W}_\varepsilon^0|_{2,\varepsilon} \leq K_0\varepsilon. \tag{2.8}$$

Then, for any $T > 0$, there exists $\varepsilon_0(T) > 0$ and $M_0(T) > 0$ such that, for any $\varepsilon < \varepsilon_0$, the unique solution $(\tilde{\varphi}_\varepsilon, \tilde{U}_\varepsilon)$ of problem (1.6)-(1.9) *in Ω_ε satisfies*

$$\sup_{t\in[0,T]}\left(|\tilde{\varphi}_\varepsilon - \tilde{\varphi}_0 - \varepsilon\tilde{\varphi}_1|_{2,\varepsilon} + |\tilde{U}_\varepsilon - \tilde{U}_0|_{2,\varepsilon}\right) \leq M_0(T)\varepsilon,$$

$$\sup_{t\in[0,T]}\left(|\nabla(\tilde{\varphi}_\varepsilon - \tilde{\varphi}_0 - \varepsilon\tilde{\varphi}_1)|_{2,\varepsilon}\right) \leq M_0(T)\varepsilon^{\frac{1}{2}}.$$

Remark 2.1
The functions $\tilde{\varphi}_1^0$ and $\tilde{\varphi}_1$ depend of course on ε implicitly because they are defined on Ω_ε. This dependence will never appear explicitly in the remaining of the paper, in order to simplify the notations.

- **Non-metastable homogeneous initial mixture $\tilde{\varphi}_0 \equiv \omega$, with $F''(\omega) \leq 0$:**

In this context, the asymptotic behavior of the solution when ε goes to zero can be precised, as we can justify the validity in times of order $O(1)$ of the asymptotic expansion

$$\tilde{\varphi}_\varepsilon \equiv \tilde{\varphi}_0 + \varepsilon \tilde{\varphi}_1 + o(\varepsilon).$$

Hence, the evolution of the small initial perturbation is well described at order ε by the solution $\tilde{\varphi}_1$ of the linearized equation.

We will see in the proof of the following result, that the main difference, compared to the previous case, lies in the fact that there is no important capillary forces in this case on $O(1)$ times. Indeed, a homogeneous solution of the Cahn-Hilliard equation does not present interfaces, and therefore, there is no capillary forces. This fact subsists when we apply a small perturbation on the solution, until an interface is created. Hence, the validity of the asymptotic expansion is only controlled by the destabilization of the non-metastable state $\tilde{\varphi}_0 = \omega$ for the 1D Cahn-Hilliard equation.

Theorem 2.2
Let us consider a constant solution $\tilde{\varphi}_0 = \varphi_0^0 \equiv \omega$ of the 1D Cahn-Hilliard equation (2.1)-(2.2)*, with $F''(\omega) \leq 0$ (e.g. ω is not metastable). Then, let us consider a family of initial data of the form*

$$\tilde{\varphi}_\varepsilon(0) = \varphi_0^0 + \varepsilon \tilde{\varphi}_1^0 = \omega + \varepsilon \tilde{\varphi}_1^0,$$

$$\tilde{U}_\varepsilon(0) = u_0^0(z) e_x + \tilde{W}_\varepsilon^0,$$

for system (1.6)-(1.9) *in the domain Ω_ε. We suppose that $m(\tilde{\varphi}_1^0) = 0$ and that there exists a constant K_0 depending only on $\varphi_0^0 = \omega$ such that*

$$|\tilde{\varphi}_1^0|_{2,\varepsilon} + \varepsilon^{\frac{1}{2}} |\nabla \tilde{\varphi}_1^0|_{2,\varepsilon} \leq K_0, \tag{2.9}$$

$$|\tilde{W}_\varepsilon^0|_{2,\varepsilon} \leq K_0 \varepsilon^{\frac{5}{4}}. \tag{2.10}$$

Then, for any $T > 0$, there exists $\varepsilon_0(T) > 0$ and $M_0(T) > 0$ such that, for any $\varepsilon < \varepsilon_0$, the solution $(\tilde{\varphi}_\varepsilon, \tilde{U}_\varepsilon)$ of (1.6)-(1.9) *in Ω_ε satisfies*

$$\sup_{t \in [0,T]} \left(|\tilde{\varphi}_\varepsilon - \tilde{\varphi}_0 - \varepsilon \tilde{\varphi}_1|_{2,\varepsilon} + |\tilde{U}_\varepsilon - \tilde{U}_0|_{2,\varepsilon} \right) \leq M_0(T) \varepsilon^{\frac{5}{4}},$$

$$\sup_{t \in [0,T]} \left(|\nabla(\tilde{\varphi}_\varepsilon - \tilde{\varphi}_0 - \varepsilon \tilde{\varphi}_1)|_{2,\varepsilon} \right) \leq M_0(T) \varepsilon^{\frac{3}{4}}.$$

- **Metastable homogeneous initial mixture $\tilde{\varphi}_0 \equiv \omega$, with $F''(\omega) > 0$:**

If we make the assumption that the 1D solution under study is an homogeneous mixture which lies in the metastability domain of the Cahn-Hilliard potential F, then we prove the persistency of perturbations of size ε all along the time, and that the asymptotic expansion

$$\tilde{\varphi}_\varepsilon = \tilde{\varphi}_0 + \varepsilon \tilde{\varphi}_1 + o(\varepsilon),$$

is justified uniformly in time.

The intuitive explanation for this result is that, when $\varphi_0 = \omega$ is a metastable state for the Cahn-Hilliard equation, it is a stable mixture for the 1D Cahn-Hilliard equation, so that a small perturbation of this state does not create an interface and so there is no significant capillary forces in the system at any time, which would have destabilized the velocity field and thus the mixture composition.

Theorem 2.3
Let us consider a constant solution $\tilde{\varphi}_0 = \varphi_0^0 \equiv \omega$ of the 1D Cahn-Hilliard equation (2.1)-(2.2)*, with $F''(\omega) > 0$, that is to say that ω is a metastable state of the potential F. Then, we consider a family of initial data*

$$\tilde{\varphi}_\varepsilon(0) = \varphi_0^0 + \varepsilon \tilde{\varphi}_1^0 = \omega + \varepsilon \tilde{\varphi}_1^0,$$

$$\tilde{U}_\varepsilon(0) = u_0^0(z)e_x + \tilde{W}_\varepsilon^0,$$

for equations (1.6)-(1.9) *in* Ω_ε. *We suppose that* $m(\tilde{\varphi}_1^0) = 0$ *and that there exists a constant* K_0 *depending only on* $\varphi_0^0 = \omega$ *such that*

$$|\tilde{\varphi}_1^0|_{2,\varepsilon} + \varepsilon^{\frac{1}{2}}|\nabla\tilde{\varphi}_1^0|_{2,\varepsilon} \leq K_0, \tag{2.11}$$

$$|\tilde{W}_\varepsilon^0|_{2,\varepsilon} \leq K_0\varepsilon^{\frac{5}{4}}. \tag{2.12}$$

There exists $\varepsilon_0 > 0$ *and* $M_0 > 0$ *such that, for any* $\varepsilon < \varepsilon_0$, *the solution* $(\tilde{\varphi}_\varepsilon, \tilde{U}_\varepsilon)$ *of* (1.6)-(1.9) *in* Ω_ε *satisfies*

$$\sup_{t\in\mathbb{R}^+}\left(|\tilde{\varphi}_\varepsilon - \tilde{\varphi}_0 - \varepsilon\tilde{\varphi}_1|_{2,\varepsilon} + |\tilde{U}_\varepsilon - \tilde{U}_0|_{2,\varepsilon}\right) \leq M_0\,\varepsilon^{\frac{5}{4}},$$

$$\sup_{t\in\mathbb{R}^+}\left(|\nabla(\tilde{\varphi}_\varepsilon - \tilde{\varphi}_0 - \varepsilon\tilde{\varphi}_1)|_{2,\varepsilon}\right) \leq M_0\,\varepsilon^{\frac{3}{4}}.$$

From now on, we are concerned with the proofs of these results. The idea is to find out the equations satisfied by the remainders of the asymptotic expansion. Then, we perform some accurate energy estimates on these remainders which allow us to conclude in the two first cases. For the proof of the third theorem, it is necessary to make some of these estimates more precise, that is the reason why it is given in a separate section at the end of this paper.

3 Preliminaries

3.1 One-dimensional solutions of the system

In the sequel, we consider a given solution $(t,z) \mapsto \tilde{\varphi}_0(t,z)$ in $\mathbb{R}\times]0,1[$ of the 1D Cahn-Hilliard equation (2.1)-(2.2) that we recall here for convenience:

$$\partial_t\tilde{\varphi}_0 - \partial_z^2\tilde{\mu}_0 = 0,$$
$$\tilde{\mu}_0 = -\alpha^2\partial_z^2\tilde{\varphi}_0 + F'(\tilde{\varphi}_0),$$

with initial data

$$\tilde{\varphi}_0(0) = \varphi_0^0 \in H^1(]0,1[),$$

and boundary conditions

$$\partial_z\tilde{\varphi}_0 = \partial_z^3\tilde{\varphi}_0 = 0, \text{ for } z = 0 \text{ and } z = 1.$$

Existence and uniqueness of such a solution is classical [1, 22], and we have the following properties

$$\forall t \geq 0,\;\; m(\tilde{\varphi}_0(t)) = m(\varphi_0^0),$$

$$\forall t \geq 0,\;\; |\partial_z\tilde{\varphi}_0|_2^2(t) + \int_{-1}^{1} F(\tilde{\varphi}_0(t,z))dz \leq C_0, \tag{3.1}$$

$$\forall T \geq 0,\;\; \int_0^T |\partial_z^3\tilde{\varphi}_0|_2^2 dt \leq C_0(1+T), \tag{3.2}$$

$$\sup_{t,z}|\tilde{\varphi}_0(t,z)| \leq C_0, \tag{3.3}$$

for a constant C_0 depending only on φ_0^0. Let us remark that the boundary condition $\partial_z^3\tilde{\varphi}_0 = 0$ makes a sense only if the solution is more regular.

We associate to this particular solution the pressure field $\tilde{p}_0$ given by (2.3) as well as the solution of the one-dimensional heat equation

$$\partial_t \tilde{u}_0 - \partial_z^2 \tilde{u}_0 = 0, \tag{3.4}$$

with initial data

$$\tilde{u}_0(0) = u_0^0 \in L^2(]0,1[),$$

and homogeneous Dirichlet boundary conditions

$$\tilde{u}_0(z=0) = \tilde{u}_0(z=1) = 0.$$

Once again, existence and uniqueness for this Cauchy problem are classical and we only recall the following estimates:

$$\|\tilde{u}_0\|_{L^\infty(\mathbb{R}^+, L^2(]0,1[))} + \|\tilde{u}_0\|_{L^2(\mathbb{R}^+, H^1(]0,1[))} \le C|u_0^0|_2. \tag{3.5}$$

Hence, by Sobolev embedding $H^1 \subset L^\infty$ in 1D, it follows

$$\|\tilde{u}_0\|_{L^2(\mathbb{R}^+, L^\infty(]0,1[))} \le C|u_0^0|_2. \tag{3.6}$$

3.2 Change of variables

In order to take into account the fact that the domain Ω_ε is very stretched in the x-direction, we naturally use a change of variables which let us perform the estimates in a domain which is independent of ε.

Namely, for any $(x,z) \in \Omega_0 =]0,1[\times]0,1[$, we introduce

$$\varphi_\varepsilon(t,x,z) = \tilde{\varphi}_\varepsilon(t, \frac{x}{\varepsilon}, z), \qquad \mu_\varepsilon(t,x,z) = \tilde{\mu}_\varepsilon(t, \frac{x}{\varepsilon}, z),$$

$$U_\varepsilon(t,x,z) = \begin{pmatrix} \tilde{u}_\varepsilon(t, \frac{x}{\varepsilon}, z) \\ \frac{1}{\varepsilon}\tilde{v}_\varepsilon(t, \frac{x}{\varepsilon}, z) \end{pmatrix}, \qquad p_\varepsilon(t,x,z) = \tilde{p}_\varepsilon(t, \frac{x}{\varepsilon}, z).$$

The coefficient $\frac{1}{\varepsilon}$ in the definition of the second coordinate of U_ε is natural to ensure that the divergence-free condition in Ω_0 for the velocity field U_ε always reads

$$\begin{pmatrix} \partial_x \\ \partial_z \end{pmatrix} . U_\varepsilon = 0.$$

Remark 3.1
As we have seen that the one-dimensional solutions $(\tilde{\varphi}_0, \tilde{U}_0)$ of the system satisfy $\tilde{v}_0 = 0$, it is easily seen that they are invariant through the previous change of variables.

Consequently, from now on, these solutions will be indifferently denoted by $(\tilde{\varphi}_0, \tilde{U}_0)$ or (φ_0, U_0).

A straightforward calculation shows that, in the new variables, system (1.6)-(1.9) reads

$$\partial_t \varphi_\varepsilon + z\partial_x \varphi_\varepsilon + \varepsilon U_\varepsilon . \nabla \varphi_\varepsilon - \varepsilon^2 \partial_x^2 \mu_\varepsilon - \partial_z^2 \mu_\varepsilon = 0, \tag{3.7}$$

$$\mu_\varepsilon = -\alpha^2 \varepsilon^2 \partial_x^2 \varphi_\varepsilon - \alpha^2 \partial_z^2 \varphi_\varepsilon + F'(\varphi_\varepsilon), \tag{3.8}$$

$$\partial_t U_\varepsilon - \varepsilon^2 \partial_x^2 U_\varepsilon - \partial_z^2 U_\varepsilon + \begin{pmatrix} \varepsilon \partial_x p_\varepsilon \\ \frac{1}{\varepsilon}\partial_z p_\varepsilon \end{pmatrix} = \begin{pmatrix} -\alpha^2 \varepsilon(\varepsilon^2 \partial_x^2 \varphi_\varepsilon + \partial_z^2 \varphi_\varepsilon)\partial_x \varphi_\varepsilon \\ \frac{-\alpha^2}{\varepsilon}(\varepsilon^2 \partial_x^2 \varphi_\varepsilon + \partial_z^2 \varphi_\varepsilon)\partial_z \varphi_\varepsilon \end{pmatrix}, \tag{3.9}$$

$$\text{div } (U_\varepsilon) = 0, \tag{3.10}$$

associated with the periodicity conditions in the x-variable and with the boundary conditions

$$\partial_z \varphi_\varepsilon = \partial_z^3 \varphi_\varepsilon = 0 \text{ on } \{z=0\} \cup \{z=1\},$$

$$U_\varepsilon = 0 \text{ on } \{z = 0\} \cup \{z = 1\}.$$

Indeed, as $\partial_z \varphi_\varepsilon = 0$ for regular enough solutions, the Neumann conditions for μ_ε on the boundaries $\{z = 0\}$ and $\{z = 1\}$ can be expressed in the following way:

$$0 = \partial_z(-\alpha^2\varepsilon^2\partial_x^2\varphi_\varepsilon - \alpha^2\partial_z^2\varphi_\varepsilon) + F''(\varphi_\varepsilon)\partial_z\varphi_\varepsilon = -\alpha^2\partial_z^3\varphi_\varepsilon.$$

3.3 Anisotropic Sobolev inequalities

The initial domain Ω_ε depends on ε. We saw that it is important to rewrite the equations and the computations in a fixed domain Ω_0. Indeed, the various constants which may appear in the energy estimates will not depend on ε. Nevertheless, if we only use classical Sobolev inequalities, we miss the fact that the physical domain is indeed anisotropic, which can prevent us from obtaining interesting results.

That is the reason why, in the following, we need to use anisotropic Sobolev inequalities (see [23]) that we recall in this section.

• **Notation:** In the sequel, $|.|_p$ represents the classical $L^p(\Omega_0)$ norm.

Lemma 3.1 (Anisotropic Sobolev inequalities)

• *For any p,* $2 \le p < +\infty$*, there exists a constant* $C_p > 0$ *such that for any function* f *in* $H^1(\Omega_0)$*, we have*

$$|f|_p \le C_p |f|_2^{\frac{2}{p}} (|f|_2 + |\partial_x f|_2)^{\frac{1}{2}-\frac{1}{p}} (|f|_2 + |\partial_z f|_2)^{\frac{1}{2}-\frac{1}{p}}. \tag{3.11}$$

• *For any p,* $2 \le p \le \infty$*, there exists a constant* $C_p > 0$ *such that for any function* f *in* $H^2(\Omega_0)$*, we have*

$$|f|_p \le C_p |f|_2^{\frac{1}{2}+\frac{1}{p}} (|f|_2 + |\partial_x f|_2 + |\partial_x^2 f|_2)^{\frac{1}{4}-\frac{1}{2p}} (|f|_2 + |\partial_z f|_2 + |\partial_z^2 f|_2)^{\frac{1}{4}-\frac{1}{2p}}. \tag{3.12}$$

Remark 3.2
In the remaining of this work, inequality (3.12) *is always applied to functions* f *satisfying periodicity conditions and homogeneous (Dirichlet or Neumann) conditions on the boundaries* $\{z = 0\}$ *and* $\{z = 1\}$ *of* Ω_0*. Then, by integrating by parts, we can show that for such functions*

$$|\partial_x f|_2 \le |f|_2^{\frac{1}{2}} |\partial_x^2 f|_2^{\frac{1}{2}} \le \frac{1}{2}(|f|_2 + |\partial_x^2 f|_2),$$

$$|\partial_z f|_2 \le |f|_2^{\frac{1}{2}} |\partial_z^2 f|_2^{\frac{1}{2}} \le \frac{1}{2}(|f|_2 + |\partial_z^2 f|_2),$$

so that (3.12) *can be used in the following form:*

$$|f|_p \le C_p |f|_2^{\frac{1}{2}+\frac{1}{p}} (|f|_2 + |\partial_x^2 f|_2)^{\frac{1}{4}-\frac{1}{2p}} (|f|_2 + |\partial_z^2 f|_2)^{\frac{1}{4}-\frac{1}{2p}}. \tag{3.13}$$

3.4 Two-dimensional solution of the linearized equation

We are now concerned with the resolution of the 2D linearized equation (2.5)-(2.6). In the domain Ω_0, this equation reads

$$\partial_t\varphi_1 + z\partial_x\varphi_1 + \varepsilon u_0(t,z)\partial_x\varphi_1 - \varepsilon^2\partial_x^2\mu_1 - \partial_z^2\mu_1 = 0, \tag{3.14}$$
$$\mu_1 = -\alpha^2\varepsilon^2\partial_x^2\varphi_1 - \alpha^2\partial_z^2\varphi_1 + F''(\varphi_0(t,z))\varphi_1, \tag{3.15}$$

where φ_1 and μ_1 are obtained from $\tilde{f}y_1$ and $\tilde{\mu}_1$ through the change of variables introduced in section 3.2.

The existence and uniqueness of a global weak solution for this problem is proven without difficulties for a given initial data φ_1^0 in $H^1(\Omega_0)$ (see [6] for instance). Let us now give some useful estimates on this solution.

In the remaining of this paper, C_0 stands for a constant that only depends on φ_0 and u_0 (that is to say on φ_0^0 and u_0^0), and C a universal constant. These constants may eventually change from line to line.

Lemma 3.2 (Regularity results and bounds for φ_1)

- *The 1D solution* (φ_0, u_0) *of the problem being fixed, the unique solution of* (3.14)-(3.15) *satisfies, for any* $T > 0$*,*

$$\forall 0 \le t \le T, \;\; |\varphi_1(t)|_2^2 \le |\varphi_1^0|^2 e^{C_0 T}, \tag{3.16}$$

$$\forall 0 \le t \le T, \;\; |\partial_x \varphi_1(t)|_2^2 \le |\partial_x \varphi_1^0|_2^2 e^{C_0 T}, \tag{3.17}$$

$$\forall 0 \le t \le T, \;\; |\partial_z \varphi_1(t)|_2^2 \le C_0 \left(|\partial_z \varphi_1^0|_2^2 + \frac{1}{\varepsilon} |\varphi_1^0|_2^2 \right) e^{C_0 T}, \tag{3.18}$$

$$\int_0^T \left(\varepsilon^4 |\partial_x^2 \varphi_1|_2^2 + \varepsilon^2 |\partial_x \partial_z \varphi_1|_2^2 + |\partial_z^2 \varphi_1|_2^2 \right) dt \le C_0 |\varphi_1^0|_2^2 e^{C_0 T}, \tag{3.19}$$

$$\int_0^T \left(\varepsilon^4 |\partial_x^3 \varphi_1|_2^2 + \varepsilon^2 |\partial_x^2 \partial_z \varphi_1|_2^2 + |\partial_x \partial_z^2 \varphi_1|_2^2 \right) dt \le C_0 |\partial_x \varphi_1^0|_2^2 e^{C_0 T}, \tag{3.20}$$

$$\int_0^T \left(\varepsilon^4 |\partial_x^2 \partial_z \varphi_1|_2^2 + 2\varepsilon^2 |\partial_x \partial_z^2 \varphi_1|_2^2 + |\partial_z^3 \varphi_1|_2^2 \right) dt \le C_0 \left(|\partial_z \varphi_1^0|_2^2 + \frac{1}{\varepsilon} |\varphi_1^0|_2^2 \right) e^{C_0 T}. \tag{3.21}$$

- *Moreover, if we suppose that* φ_0 *is a constant which is a metastable state of the Cahn-Hilliard potential, that is to say if* $\varphi_0(t,z) = \omega$ *for any* (t,z) *with* $F''(\omega) > 0$*, then the estimates on* φ_1 *read*

$$\forall t \ge 0, \;\; |\varphi_1(t)|_2^2 \le |\varphi_1^0|_2^2,$$

$$\forall t \ge 0, \;\; |\partial_x \varphi_1(t)|_2^2 \le |\partial_x \varphi_1^0|_2^2,$$

$$\forall t \ge 0, \;\; |\partial_z \varphi_1(t)|_2^2 \le |\partial_z \varphi_1^0|_2^2 + \frac{C_0}{\varepsilon} |\varphi_1^0|_2^2,$$

$$\int_0^T \left(\varepsilon^2 |\partial_x \varphi_1|_2^2 + |\partial_z \varphi_1|_2^2 + \varepsilon^4 |\partial_x^2 \varphi_1|_2^2 + \varepsilon^2 |\partial_x \partial_z \varphi_1|_2^2 + |\partial_z^2 \varphi_1|_2^2 \right) dt \le C_0 |\varphi_1^0|_2^2,$$

$$\int_0^T \left(\varepsilon^2 |\partial_x^2 \varphi_1|_2^2 + |\partial_z^2 \varphi_1|_2^2 + \varepsilon^4 |\partial_x^3 \varphi_1|_2^2 + \varepsilon^2 |\partial_x^2 \partial_z \varphi_1|_2^2 + |\partial_x \partial_z^2 \varphi_1|_2^2 \right) dt \le C_0 |\partial_x \varphi_1^0|_2^2,$$

$$\int_0^T \left(\varepsilon^2 |\partial_x \partial_z \varphi_1|_2^2 + |\partial_z^2 \varphi_1|_2^2 + \varepsilon^4 |\partial_x^2 \partial_z \varphi_1|_2^2 + 2\varepsilon^2 |\partial_x \partial_z^2 \varphi_1|_2^2 + |\partial_z^3 \varphi_1|_2^2 \right) dt \le C_0 \left(|\partial_z \varphi_1^0|_2^2 + \frac{1}{\varepsilon} |\varphi_1^0|_2^2 \right).$$

Proof :

- ***Step 1:***

We take the inner product of (3.14) by $(-1)^k \partial_x^{2k} \varphi_1$ (for $k = 0$ or 1). Thanks to the periodicity in the x-direction, we obtain, after integrations by parts,

$$\frac{1}{2}\frac{d}{dt} |\partial_x^k \varphi_1|_2^2 + \alpha^2 \varepsilon^4 |\partial_x^{k+2} \varphi_1|_2^2 + 2\alpha^2 \varepsilon^2 |\partial_x^{k+1} \partial_z \varphi_1|_2^2 + \alpha^2 |\partial_x^k \partial_z^2 \varphi_1|_2^2 \le |F''(\varphi_0)|_\infty |\partial_x^k \varphi_1|_2 (\varepsilon^2 |\partial_x^{k+2} \varphi_1|_2 + |\partial_x^k \partial_z^2 \varphi_1|_2),$$

because φ_0 and u_0 are independent of x. This implies, by Gronwall's lemma

$$\forall 0 \le t \le T, \;\; |\partial_x^k \varphi_1|_2^2 \le |\partial_x^k \varphi_1^0|_2^2 e^{C_0 T},$$

with $C_0 = \frac{1}{\alpha^2} \sup_{(t,z)} |F''(\varphi_0(t,z))|^2$ (which is finite as φ_0 belongs to $L^\infty(\mathbb{R}^+ \times \Omega)$). Furthermore, one has

$$\int_0^T \left(\varepsilon^4 |\partial_x^{k+2} \varphi_1|_2^2 + \varepsilon^2 |\partial_x^{k+1} \partial_z \varphi_1|_2^2 + |\partial_x^k \partial_z^2 \varphi_1|_2^2 \right) dt \le C_0 |\partial_x^k \varphi_1^0|_2^2 e^{C_0 T}.$$

- ***Step 2:***

If we take the inner product of (3.14) by $-\partial_z^2\varphi_1$, we have after integrations by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_z\varphi_1|_2^2 &+ \alpha^2\varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + 2\alpha^2\varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + \alpha^2|\partial_z^3\varphi_1|_2^2 \\ &\leq \left|\int_{\Omega_0}\varphi_1\partial_x\partial_z\varphi_1 dx\right| + \varepsilon\left|\int_{\Omega_0}u_0\varphi_1\partial_x\partial_z^2\varphi_1 dx\right| \\ &\quad + |F''(\varphi_0)|_\infty|\partial_z\varphi_1|_2(\varepsilon^2|\partial_x^2\partial_z\varphi_1|_2 + |\partial_z^3\varphi_1|_2) \\ &\quad + |F'''(\varphi_0)\partial_z\varphi_0|_\infty|\varphi_1|_2(\varepsilon^2|\partial_x^2\partial_z\varphi_1|_2 + |\partial_z^3\varphi_1|_2),\end{aligned}$$

so that, if we use the coercive terms to absorb some of the right-hand side member terms, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_z\varphi_1|_2^2 &+ \alpha^2\varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + \alpha^2\varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + \alpha^2|\partial_z^3\varphi_1|_2^2 \\ &\leq \frac{C_0}{\varepsilon}(|\varphi_1|_2^2 + \varepsilon^2|\partial_x\partial_z\varphi_1|_2^2) + C|u_0|_\infty^2|\varphi_1|_2^2 + C_0|\partial_z\varphi_1|_2^2.\end{aligned}$$

Integrating this inequality on $[0,t]$, and thanks to (3.5) and the previous estimates on φ_1, we obtain

$$\begin{aligned}\forall t\in[0,T],\ \ |\partial_z\varphi_1(t)|_2^2 &+ \alpha^2\int_0^t(\varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + \varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + |\partial_z^3\varphi_1|_2^2)dt \\ &\leq |\partial_z\varphi_1^0|_2^2 + \frac{C_0}{\varepsilon}\left(\int_0^T|\varphi_1|_2^2dt + \varepsilon^2\int_0^T|\partial_x\partial_z\varphi_1|_2^2dt\right) \\ &\quad + C\int_0^T|\partial_z u_0|_2^2|\varphi_1|_2^2dt + C_0\int_0^t|\partial_z\varphi_1|_2^2dt \\ &\leq |\partial_z\varphi_1^0|_2^2 + \frac{C_0}{\varepsilon}|\varphi_1^0|_2^2e^{C_0T} + C|\varphi_1^0|_2^2|u_0^0|_2^2e^{C_0T} + C_0\int_0^t|\partial_z\varphi_1|_2^2dt.\end{aligned}$$

Applying one more time the Gronwall's lemma, one has

$$\begin{aligned}\forall t\in[0,T],\ \ |\partial_z\varphi_1|_2^2 &\leq \left(|\partial_z\varphi_1^0|_2^2 + \frac{C_0}{\varepsilon}|\varphi_1^0|_2^2e^{C_0T}\right)e^{C_0T} \\ &\leq C_0\left(|\partial_z\varphi_1^0|_2^2 + \frac{1}{\varepsilon}|\varphi_1^0|_2^2\right)e^{C_0T},\end{aligned}$$

and

$$\begin{aligned}\int_0^T\left(\varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + 2\varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + |\partial_z^3\varphi_1|_2^2\right)dt &\leq C_0\left(|\partial_z\varphi_1^0|_2^2 + \frac{1}{\varepsilon}|\varphi_1^0|_2^2e^{C_0T}\right)e^{C_0T} \\ &\leq C_0\left(|\partial_z\varphi_1^0|_2^2 + \frac{1}{\varepsilon}|\varphi_1^0|_2^2\right)e^{C_0T}.\end{aligned}$$

• ***Step 3:***

In the particular case of a reference solution φ_0 which is a metastable constant solution the term $F''(\varphi_0)\varphi_1$, which appears in the definition of μ_1, becomes a coercive term. As a consequence, the energy estimates in this case are uniform in time. More precisely, we have

$$\forall t\geq 0,\ \ |\partial_x^k\varphi_1|_2^2(t)\leq|\partial_x^k\varphi_1^0|_2^2,$$

$$\forall t\geq 0,\ \ \int_0^t\left(\varepsilon^2|\partial_x^{k+1}\varphi_1|_2^2 + |\partial_z^{k+1}\varphi_1|_2^2 + \varepsilon^4|\partial_x^{k+2}\varphi_1|_2^2 + \varepsilon^2|\partial_x^{k+1}\partial_z\varphi_1|_2^2 + |\partial_x^k\partial_z^2\varphi_1|_2^2\right)dt\leq C_0|\partial_x^k\varphi_1^0|_2^2.$$

Moreover, the estimates on the derivative of φ_1 with respect to z can be written in the following form:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_z\varphi_1|_2^2 + F''(\omega)\varepsilon^2|\partial_x\partial_z\varphi_1|_2^2 + F''(\omega)|\partial_z^2\varphi_1|_2^2 + \alpha^2\varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + \alpha^2\varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + \alpha^2|\partial_z^3\varphi_1|_2^2 \\ \leq \left|\int_{\Omega_0}\partial_x\varphi_1\partial_z\varphi_1 dx\right| + \varepsilon\left|\int_{\Omega_0}u_0\varphi_1\partial_x\partial_z^2\varphi_1 dx\right|.\end{aligned}$$

After absorbing some terms by the coercive terms of the left-hand side, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_z\varphi_1|_2^2 + F''(\omega)\varepsilon^2|\partial_x\partial_z\varphi_1|_2^2 + F''(\omega)|\partial_z^2\varphi_1|_2^2 + \alpha^2\varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + 2\alpha^2\varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + \alpha^2|\partial_z^3\varphi_1|_2^2 \\ \leq \frac{1}{\varepsilon}(\varepsilon^2|\partial_x\varphi_1|_2^2 + |\partial_z\varphi_1|_2^2) + C|u_0|_\infty^2|\varphi_1|_2^2,\end{aligned}$$

so that, integrating by parts and using (3.5) and the previous bounds on the derivatives with respect to x, one has

$$\forall t\geq 0,\;\; |\partial_z\varphi_1(t)|_2^2 \leq |\partial_z\varphi_1^0|_2^2 + \frac{C_0}{\varepsilon}|\varphi_1^0|_2^2.$$

Finally, we obtain

$$\forall t\geq 0,\;\; \int_0^t\left(\varepsilon^2|\partial_x\partial_z\varphi_1|_2^2 + |\partial_z^2\varphi_1|_2^2 + \varepsilon^4|\partial_x^2\partial_z\varphi_1|_2^2 + \varepsilon^2|\partial_x\partial_z^2\varphi_1|_2^2 + |\partial_z^3\varphi_1|_2^2\right)dt \leq |\partial_z\varphi_1^0|_2^2 + \frac{C_0}{\varepsilon}|\varphi_1^0|_2^2.$$

■

4 Energy estimates

We seek $\varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon$ and U_ε solutions of problem (3.7)-(3.10), with the following form:

$$\varphi_\varepsilon = \varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon, \tag{4.1}$$

$$\mu_\varepsilon = \mu_0 + \varepsilon\mu_1 + \varepsilon M_\varepsilon, \tag{4.2}$$

$$p_\varepsilon = p_0 + P_\varepsilon, \tag{4.3}$$

$$U_\varepsilon = U_0 + W_\varepsilon, \tag{4.4}$$

where φ_1 and μ_1 are given by the linearized equation associated to the initial data φ_1^0. Then, we obtain some bounds on the remainders R_ε and W_ε by introducing the above expansions (4.1)-(4.4) in equations (3.7)-(3.10).

4.1 Equations on the remainders

• **Equation for R_ε**

The equation for φ_ε (3.7) reads

$$\begin{aligned}\partial_t(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) + z\partial_x(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) + \varepsilon U_0.\nabla(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) + \varepsilon W_\varepsilon.\nabla(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) \\ - \varepsilon^2\partial_x^2(\mu_0 + \varepsilon\mu_1 + \varepsilon M_\varepsilon) - \partial_z^2(\mu_0 + \varepsilon\mu_1 + \varepsilon M_\varepsilon) = 0.\end{aligned}$$

Using equation (2.1) satisfied by φ_0, the one satisfied by φ_1 (3.14), the fact that $\partial_x\varphi_0 = 0$ and that U_0 is of the form (2.4), the equation satisfied by R_ε reads

$$\begin{aligned}\partial_t R_\varepsilon + z\partial_x R_\varepsilon + \varepsilon u_0\partial_x R_\varepsilon + W_\varepsilon.\nabla\varphi_0 + \varepsilon W_\varepsilon.\nabla\varphi_1 + \varepsilon W_\varepsilon.\nabla R_\varepsilon \\ - \varepsilon^2\partial_x^2 M_\varepsilon - \partial_z^2 M_\varepsilon = 0.\end{aligned} \tag{4.5}$$

Moreover, the choice of the initial condition for $\tilde{\varphi}_\varepsilon$ in the three theorems (section 2), implies that the initial data for R_ε is zero

$$R_\varepsilon(0) = 0,$$

and it is easily seen that R_ε fulfills the same boundary conditions as φ_ε, that is to say periodicity in the x-direction and Neumann boundary conditions in the z direction

$$\partial_z R_\varepsilon = 0 \text{ on } \{z=0\} \cup \{z=1\}.$$

Finally, let us remark that as $m(R_\varepsilon(0)) = 0$, the average of R_ε vanishes all along the time.

• **Equation for M_ε**

The equation satisfied by μ_ε (3.8) reads

$$(\mu_0 + \varepsilon\mu_1 + \varepsilon M_\varepsilon) = -\alpha^2\varepsilon^2\partial_x^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) - \alpha^2\partial_z^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) + F'(\varphi_0 + \varepsilon\varphi_1 + \varepsilon\varphi_\varepsilon),$$

which can be written, thanks to equations (2.2) and (3.15)

$$M_\varepsilon = -\alpha^2\varepsilon^2\partial_x^2 R_\varepsilon - \alpha^2\partial_z^2 R_\varepsilon + \frac{F'(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) - F'(\varphi_0) - \varepsilon F''(\varphi_0)\varphi_1}{\varepsilon}.$$

By Taylor's formula, we have

$$F'(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) = F'(\varphi_0) + \varepsilon F''(\varphi_0)(\varphi_1 + R_\varepsilon) + \varepsilon^2 \int_0^1 (1-s)F'''(\varphi_0 + \varepsilon s(\varphi_1 + R_\varepsilon))(\varphi_1 + R_\varepsilon)^2 ds,$$

so that, using (1.5), the equation for M_ε is finally:

$$M_\varepsilon = -\alpha^2\varepsilon^2\partial_x^2 R_\varepsilon - \alpha^2\partial_z^2 R_\varepsilon + \varepsilon F''(\varphi_0)R_\varepsilon + \varepsilon(\varphi_1 + R_\varepsilon)^2(3\varphi_0 + \varepsilon(\varphi_1 + R_\varepsilon)). \tag{4.6}$$

Moreover, thanks to the boundary conditions on μ_ε, the remainder M_ε is periodic in the x-direction and satisfies

$$\partial_z M_\varepsilon = 0, \text{ on } \{z=0\} \text{ and } \{z=1\},$$

which is equivalent, thanks to the Neumann condition on R_ε, to

$$\partial_z^3 R_\varepsilon = 0,$$

for R_ε smooth enough.

• **Equation for W_ε**

First of all, it is clear that because of (3.10) and of the form of U_0 (2.4), one has

$$\text{div } (W_\varepsilon) = \partial_x W_\varepsilon + \partial_z W_\varepsilon = 0. \tag{4.7}$$

Then, let us write the equation satisfied by U_ε and p_ε in the following way:

$$\partial_t(U_0 + W_\varepsilon) - \varepsilon^2\partial_x^2(U_0 + W_\varepsilon) - \partial_z^2(U_0 + W_\varepsilon) + \begin{pmatrix} \varepsilon\partial_x(p_0 + P_\varepsilon) \\ \frac{1}{\varepsilon}\partial_z(p_0 + P_\varepsilon) \end{pmatrix} =$$
$$\begin{pmatrix} \varepsilon^2(-\alpha^2\varepsilon^2\partial_x^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) - \alpha^2\partial_z^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon))\partial_x(\varphi_1 + R_\varepsilon) \\ \frac{1}{\varepsilon}(-\alpha^2\varepsilon^2\partial_x^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) - \alpha^2\partial_z^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon))\partial_z(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) \end{pmatrix},$$

so that using (3.4) and (2.3), there remains

$$\partial_t W_\varepsilon - \varepsilon^2\partial_x^2 W_\varepsilon - \partial_z^2 W_\varepsilon + \begin{pmatrix} \varepsilon\partial_x P_\varepsilon \\ \frac{1}{\varepsilon}\partial_z P_\varepsilon \end{pmatrix} =$$
$$\begin{pmatrix} \varepsilon^2(-\alpha^2\varepsilon^3\partial_x^2(\varphi_1 + R_\varepsilon) - \alpha^2\partial_z^2(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon))\partial_x(\varphi_1 + R_\varepsilon) \\ \\ \frac{1}{\varepsilon}(-\alpha^2\varepsilon^3\partial_x^2(\varphi_1 + R_\varepsilon)\partial_z(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) \\ -\alpha^2\varepsilon\partial_z^2(\varphi_1 + R_\varepsilon)\partial_z(\varphi_0 + \varepsilon\varphi_1 + \varepsilon R_\varepsilon) - \alpha^2\varepsilon\partial_z^2\varphi_0(\varphi_1 + R_\varepsilon)) \end{pmatrix}. \tag{4.8}$$

Furthermore, we deduce from the choice of the initial data on $\tilde{U}_\varepsilon$ in the statement of the theorems (section 2) that we have

$$W_\varepsilon(0) = W_\varepsilon^0,$$

where $W_\varepsilon^0(x,z) = \tilde{W}_\varepsilon^0(x/\varepsilon, z)$.

The boundary conditions for W_ε are the same than those on U_ε, that is to say periodicity in x and homogeneous Dirichlet condition on the physical boundary $\{z=0\} \cup \{z=1\}$.

From now on, the two coordinates of the vector field W_ε are denoted by

$$W_\varepsilon = \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}.$$

4.2 L^2 estimate on R_ε

We obtain this estimate multiplying by R_ε the equation (4.5), in which we replace M_ε by its values given by (4.6), and integrating on Ω_0

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|R_\varepsilon|_2^2 &+ \varepsilon^2 \int_{\Omega_0} \partial_x^2 \left(-\alpha^2\varepsilon^2\partial_x^2 R_\varepsilon - \alpha^2\partial_z^2 R_\varepsilon + F''(\varphi_0)R_\varepsilon + \varepsilon(\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\right) R_\varepsilon dx \\
&- \int_{\Omega_0} \partial_z^2 \left(-\alpha^2\varepsilon^2\partial_x^2 R_\varepsilon - \alpha^2\partial_z^2 R_\varepsilon + F''(\varphi_0)R_\varepsilon + \varepsilon(\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\right) R_\varepsilon dx \\
&+ \int_{\Omega_0} W_\varepsilon.\nabla\varphi_0 R_\varepsilon dx + \varepsilon \int_{\Omega_0} W_\varepsilon.\nabla\varphi_1 R_\varepsilon dx \\
&= 0.
\end{aligned}$$

We rewrite this estimate in a more convenient way:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|R_\varepsilon|_2^2 &+ \alpha^2\varepsilon^4|\partial_x^2 R_\varepsilon|_2^2 + 2\alpha^2\varepsilon^2|\partial_x\partial_z R_\varepsilon|_2^2 + \alpha^2|\partial_z^2 R_\varepsilon|_2^2 = \\
&- \varepsilon^2 \int_{\Omega_0} \partial_x(F''(\varphi_0)R_\varepsilon)\partial_x R_\varepsilon dx - \int_{\Omega_0} \partial_z(F''(\varphi_0)R_\varepsilon)\partial_z R_\varepsilon dx \\
&+ \varepsilon^3 \int_{\Omega_0} (\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\partial_x^2 R_\varepsilon dx \\
&+ \varepsilon \int_{\Omega_0} (\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\partial_z^2 R_\varepsilon dx \\
&- \int_{\Omega_0} v_\varepsilon\partial_z\varphi_0 R_\varepsilon dx - \varepsilon \int_{\Omega_0} W_\varepsilon.\nabla\varphi_1 R_\varepsilon dx.
\end{aligned} \tag{4.9}$$

4.3 L^2 estimate on W_ε

We have to use the fact that W_ε is a divergence free vector field in order to get rid of the pressure term P_ε (on which we do not have useful estimates). That is the reason why we multiply equation (4.8) in L^2 by

$$\begin{pmatrix} \dfrac{u_\varepsilon}{\varepsilon^2} \\ v_\varepsilon \end{pmatrix}.$$

In those conditions, we have

$$\int_{\Omega_0} \begin{pmatrix} \varepsilon\partial_x P_\varepsilon \\ \dfrac{1}{\varepsilon}\partial_z P_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \dfrac{u_\varepsilon}{\varepsilon^2} \\ v_\varepsilon \end{pmatrix} dx = 0.$$

The estimate reads

$$\begin{aligned}
&\frac{d}{dt}\left(\left|\frac{u_\varepsilon}{\varepsilon}\right|_2^2+|v_\varepsilon|_2^2\right)+\varepsilon^2\left|\partial_x\frac{u_\varepsilon}{\varepsilon}\right|_2^2+\left|\partial_z\frac{u_\varepsilon}{\varepsilon}\right|_2^2+\varepsilon^2|\partial_x v_\varepsilon|_2^2+|\partial_z v_\varepsilon|_2^2=\\
&\quad-\alpha^2\varepsilon\int_{\Omega_0}\partial_z^2\varphi_0\partial_x(\varphi_1+R_\varepsilon)\left(\frac{u_\varepsilon}{\varepsilon}\right)dx-\alpha^2\int_{\Omega_0}\partial_z^2\varphi_0\partial_z(\varphi_1+R_\varepsilon)v_\varepsilon dx\\
&\quad-\alpha^2\varepsilon^2\int_{\Omega_0}\partial_z^2(\varphi_1+R_\varepsilon)\partial_x(\varphi_1+R_\varepsilon)\left(\frac{u_\varepsilon}{\varepsilon}\right)dx-\alpha^2\varepsilon^4\int_{\Omega_0}\partial_x^2(\varphi_1+R_\varepsilon)\partial_x(\varphi_1+R_\varepsilon)\left(\frac{u_\varepsilon}{\varepsilon}\right)dx\\
&\quad-\alpha^2\varepsilon^2\int_{\Omega_0}\partial_x^2(\varphi_1+R_\varepsilon)\partial_z\varphi_0 v_\varepsilon dx-\alpha^2\int_{\Omega_0}\partial_z^2(\varphi_1+R_\varepsilon)\partial_z\varphi_0 v_\varepsilon dx\\
&\quad-\alpha^2\varepsilon\int_{\Omega_0}\partial_z^2(\varphi_1+R_\varepsilon)\partial_z(\varphi_1+R_\varepsilon)v_\varepsilon dx-\alpha^2\varepsilon^3\int_{\Omega_0}\partial_x^2(\varphi_1+R_\varepsilon)\partial_z(\varphi_1+R_\varepsilon)v_\varepsilon dx
\end{aligned}\tag{4.10}$$

It is clear that this estimate is useless if we do not have additional estimates on R_ε in H^1. As the problem has a strong anisotropy, we bound separately $\partial_x R_\varepsilon$ and $\partial_z R_\varepsilon$.

4.4 L^2 estimate on $\partial_x R_\varepsilon$

This estimate is obtained by multiplying equation (4.5) by $-\partial_x^2 R_\varepsilon$ and integrating by parts. As we have already seen when we established estimates on φ_1, the contribution of the transport term $(z+\varepsilon u_0)\partial_x$ is zero. Hence, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\partial_x R_\varepsilon|_2^2+\alpha^2\varepsilon^4|\partial_x^3R_\varepsilon|_2^2+\alpha^2\varepsilon^2|\partial_x^2\partial_z R_\varepsilon|_2^2+\alpha^2|\partial_x\partial_z^2R_\varepsilon|_2^2=\\
&\quad-\varepsilon^2\int_{\Omega_0}\partial_x^2(F''(\varphi_0)R_\varepsilon)\partial_x^2R_\varepsilon dx-\int_{\Omega_0}\partial_z^2(F''(\varphi_0)R_\varepsilon)\partial_x^2R_\varepsilon dx\\
&\quad+\varepsilon\int_{\Omega_0}W_\varepsilon.\nabla(\varphi_1+R_\varepsilon)\partial_x^2R_\varepsilon dx+\int_{\Omega_0}v_\varepsilon\partial_z\varphi_0\partial_x^2R_\varepsilon dx\\
&\quad-\varepsilon^3\int_{\Omega_0}\partial_x\left((\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\right)\partial_x^3R_\varepsilon dx\\
&\quad-\varepsilon\int_{\Omega_0}\partial_z\left((\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\right)\partial_z\partial_x^2R_\varepsilon dx.
\end{aligned}\tag{4.11}$$

4.5 L^2 estimate on $\partial_z R_\varepsilon$

If we multiply equation (4.5) by $-\partial_z^2 R_\varepsilon$ and integrate by parts, contrary to the previous case, the terms coming from the transport operator have a non-zero contribution. More precisely, the estimate that we obtain reads

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\partial_z R_\varepsilon|_2^2+\alpha^2\varepsilon^4|\partial_x^2\partial_z R_\varepsilon|_2^2+\alpha^2\varepsilon^2|\partial_x\partial_z^2R_\varepsilon|_2^2+\alpha^2|\partial_z^3R_\varepsilon|_2^2=\\
&\quad\int_{\Omega_0}R_\varepsilon\partial_x\partial_z R_\varepsilon dx-\varepsilon\int_{\Omega_0}\partial_z u_0 R_\varepsilon\partial_x\partial_z R_\varepsilon dx\\
&\quad-\varepsilon^2\int_{\Omega_0}\partial_x^2(F''(\varphi_0)R_\varepsilon)\partial_z^2R_\varepsilon dx-\int_{\Omega_0}\partial_z^2(F''(\varphi_0)R_\varepsilon)\partial_z^2R_\varepsilon dx\\
&\quad+\varepsilon\int_{\Omega_0}W_\varepsilon.\nabla(R_\varepsilon+\varphi_1)\partial_z^2R_\varepsilon dx+\int_{\Omega_0}v_\varepsilon\partial_z\varphi_0\partial_z^2R_\varepsilon dx\\
&\quad-\varepsilon^3\int_{\Omega_0}\partial_x\left((\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\right)\partial_x\partial_z^2R_\varepsilon dx\\
&\quad-\varepsilon\int_{\Omega_0}\partial_z\left((\varphi_1+R_\varepsilon)^2(3\varphi_0+\varepsilon(\varphi_1+R_\varepsilon))\right)\partial_z^3R_\varepsilon dx.
\end{aligned}\tag{4.12}$$

The idea is to control the first term in the right-hand side of this inequality by the coercive terms appearing in the L^2 estimate on R_ε (4.9). To achieve this, we have to multiply the previous estimate by ε and then estimate (4.11) by ε^3 to conserve the anisotropy factor of the problem and then to obtain the square of the L^2 norm of the gradient of $\tilde{R}_\varepsilon$ in the initial variables.

Let us remark that if one writes the first term of the right member of (4.12) in the following way:

$$\int_{\Omega_0} \partial_x R_\varepsilon \partial_z R_\varepsilon dx,$$

then, trying to control it only using the coercive terms in (4.11) and (4.12), it would have been necessary to multiply this estimate by ε^2 and (4.11) by ε^4. These two high powers of ε would prevent us from obtaining a final estimate which is uniform in ε.

4.6 Final estimate

We introduce the energy which is considered in the sequel:

$$y_\varepsilon(t) = \frac{1}{2}|R_\varepsilon|_2^2 + \frac{1}{2}\left|\frac{u_\varepsilon}{\varepsilon}\right|_2^2 + \frac{1}{2}|v_\varepsilon|_2^2 + \frac{1}{2}\varepsilon^3|\partial_x R_\varepsilon|_2^2 + \frac{1}{2}\varepsilon|\partial_z R_\varepsilon|_2^2. \tag{4.13}$$

Another way to introduce this energy in the initial variables is

$$y_\varepsilon(t) = |\tilde{R}_\varepsilon|_{2,\varepsilon}^2 + \left|\frac{\tilde{W}_\varepsilon}{\varepsilon}\right|_{2,\varepsilon}^2 + \varepsilon|\nabla \tilde{R}_\varepsilon|_{2,\varepsilon}^2. \tag{4.14}$$

The coercive terms associated with this energy y_ε are collected in

$$\begin{aligned} z_\varepsilon(t) &= \alpha^2\varepsilon^4|\partial_x^2 R_\varepsilon|_2^2 + 2\alpha^2\varepsilon^2|\partial_x\partial_z R_\varepsilon|_2^2 + \alpha^2|\partial_z^2 R_\varepsilon|_2^2 \\ &\quad + \alpha^2\varepsilon^7|\partial_x^3 R_\varepsilon|_2^2 + \alpha^2\varepsilon^5|\partial_x^2\partial_z R_\varepsilon|_2^2 + \alpha^2\varepsilon^3|\partial_x\partial_z^2 R_\varepsilon|_2^2 + \alpha^2\varepsilon|\partial_z^3 R_\varepsilon|_2^2 \\ &\quad + \varepsilon^2\left|\partial_x\left(\frac{u_\varepsilon}{\varepsilon}\right)\right|_2^2 + \left|\partial_z\left(\frac{u_\varepsilon}{\varepsilon}\right)\right|_2^2 + \varepsilon^2|\partial_x v_\varepsilon|_2^2 + |\partial_z v_\varepsilon|_2^2. \end{aligned}$$

Therefore, the final estimate is obtained by summing (4.9), (4.10), $\varepsilon\times$(4.12) and $\varepsilon^3\times$(4.11),

$$\begin{aligned} \frac{d}{dt}y_\varepsilon + z_\varepsilon &= I_1 + ... + I_6 \\ &\quad + J_1 + ... + J_8 \\ &\quad + K_1 + ... + K_6 \\ &\quad + L_1 + ... + L_8, \end{aligned}$$

where $I_1, ..., I_6$ stand for the six integrals in the second member of (4.9), $J_1, ..., J_8$ for the eight integrals in estimate (4.9), $K_1, ..., K_6$ for the six integrals of (4.11) multiplied by ε^3 and finally $L_1, ..., L_8$ stand for the eight terms of the right-hand side of (4.12) multiplied by ε.

We just have now to evaluate those twenty-eight integrals in terms of y_ε and z_ε.

First of all, let us remark that, the average of R_ε being zero, the properties of regularity of the Laplace operator with Neumann boundary conditions give

$$|R_\varepsilon|_2 \leq C(|\partial_x^2 R_\varepsilon|_2 + |\partial_z^2 R_\varepsilon|_2). \tag{4.15}$$

Moreover, the Poincaré inequality for the functions whose average vanishes in Ω_0 leads to

$$|R_\varepsilon|_2 \leq C(|\partial_x R_\varepsilon|_2 + |\partial_z R_\varepsilon|_2). \tag{4.16}$$

- ***Step 1 - Terms issued from the L^2 estimate on R_ε***

We integrate by parts the first two terms to obtain

$$
\begin{aligned}
|I_1| &\leq \varepsilon^2 |F''(\varphi_0)|_\infty |R_\varepsilon|_2 |\partial_x^2 R_\varepsilon|_2 \leq \frac{\alpha^2}{50}\varepsilon^4 |\partial_x^2 R_\varepsilon|_2^2 + C|F''(\varphi_0)|_\infty^2 |R_\varepsilon|_2^2 \\
&\leq \frac{1}{50} z_\varepsilon + C|F''(\varphi_0)|_\infty^2 y_\varepsilon,
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
|I_2| &\leq |F''(\varphi_0)|_\infty |R_\varepsilon|_2 |\partial_z^2 R_\varepsilon|_2 \leq \frac{\alpha^2}{50} |\partial_z^2 R_\varepsilon|_2^2 + C|F''(\varphi_0)|_\infty^2 |R_\varepsilon|_2^2 \\
&\leq \frac{1}{50} z_\varepsilon + C|F''(\varphi_0)|_\infty^2 y_\varepsilon.
\end{aligned}
\tag{4.18}
$$

Using the anisotropic Sobolev inequality (3.13) with $p = 4$, we obtain

$$
\begin{aligned}
\varepsilon^{\frac{1}{4}} |R_\varepsilon|_4 &\leq C|R_\varepsilon|_2^{\frac{3}{4}} (\varepsilon^2 |R_\varepsilon|_2 + \varepsilon^2 |\partial_x^2 R_\varepsilon|_2)^{\frac{1}{8}} (|R_\varepsilon|_2 + |\partial_z^2 R_\varepsilon|_2)^{\frac{1}{8}} \\
&\leq C y_\varepsilon^{\frac{3}{8}} (y_\varepsilon^{\frac{1}{2}} + z_\varepsilon^{\frac{1}{2}})^{\frac{1}{4}} \leq C y_\varepsilon^{\frac{3}{8}} (y_\varepsilon^{\frac{1}{8}} + z_\varepsilon^{\frac{1}{8}}),
\end{aligned}
\tag{4.19}
$$

and using the anisotropic Sobolev inequality (3.13) with $p = 6$, we have

$$
\begin{aligned}
\varepsilon^{\frac{1}{3}} |R_\varepsilon|_6 &\leq C|R_\varepsilon|_2^{\frac{2}{3}} (\varepsilon^2 |R_\varepsilon|_2 + \varepsilon^2 |\partial_x^2 R_\varepsilon|_2)^{\frac{1}{6}} (|R_\varepsilon|_2 + |\partial_z^2 R_\varepsilon|_2)^{\frac{1}{6}} \\
&\leq C y_\varepsilon^{\frac{1}{3}} (y_\varepsilon^{\frac{1}{2}} + z_\varepsilon^{\frac{1}{2}})^{\frac{1}{3}} \\
&\leq C y_\varepsilon^{\frac{1}{3}} (y_\varepsilon^{\frac{1}{6}} + z_\varepsilon^{\frac{1}{6}}).
\end{aligned}
\tag{4.20}
$$

Hence, thanks to the two previous estimates, one has

$$
\begin{aligned}
|I_3| &\leq \varepsilon^3 |\varphi_0|_\infty |\varphi_1|_4^2 |\partial_x^2 R_\varepsilon|_2 + \varepsilon^3 |\varphi_0|_\infty |R_\varepsilon|_4^2 |\partial_x^2 R_\varepsilon|_2 + \varepsilon^4 |\varphi_1|_6^3 |\partial_x^2 R_\varepsilon|_2 + \varepsilon^4 |R_\varepsilon|_6^3 |\partial_x^2 R_\varepsilon|_2 \\
&\leq \varepsilon |\varphi_0|_\infty |\varphi_1|_4^2 (\varepsilon^2 |\partial_x^2 R_\varepsilon|_2) + \varepsilon^{\frac{1}{2}} |\varphi_0|_\infty (\varepsilon^{\frac{1}{4}} |R_\varepsilon|_4)^2 (\varepsilon^2 |\partial_x^2 R_\varepsilon|_2) \\
&\qquad + \varepsilon^2 |\varphi_1|_6^3 (\varepsilon^2 |\partial_x^2 R_\varepsilon|_2) + \varepsilon (\varepsilon^{\frac{1}{3}} |R_\varepsilon|_6)^3 (\varepsilon^2 |\partial_x^2 R_\varepsilon|_2) \\
&\leq \varepsilon |\varphi_0|_\infty |\varphi_1|_4^2 z_\varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{1}{2}} |\varphi_0|_\infty y_\varepsilon^{\frac{3}{4}} (y_\varepsilon^{\frac{1}{4}} + z_\varepsilon^{\frac{1}{4}}) z_\varepsilon^{\frac{1}{2}} + \varepsilon^2 |\varphi_1|_6^3 z_\varepsilon^{\frac{1}{2}} + C\varepsilon y_\varepsilon (y_\varepsilon^{\frac{1}{2}} + z_\varepsilon^{\frac{1}{2}}) z_\varepsilon^{\frac{1}{2}} \\
&\leq \frac{1}{50} z_\varepsilon + C\varepsilon^2 |\varphi_0|_\infty^2 |\varphi_1|_4^4 + C\varepsilon^4 |\varphi_1|_6^6 + C\varepsilon |\varphi_0|_\infty^2 y_\varepsilon^2 + C\varepsilon^2 (1 + |\varphi_0|_\infty^4) y_\varepsilon^3 + C\varepsilon y_\varepsilon z_\varepsilon.
\end{aligned}
\tag{4.21}
$$

The following term can be treated in the same way, replacing formally $\varepsilon^2 \partial_x^2 R_\varepsilon$ by $\partial_z^2 R_\varepsilon$,

$$
|I_4| \leq \frac{1}{50} z_\varepsilon + C\varepsilon^2 |\varphi_0|_\infty^2 |\varphi_1|_4^4 + C\varepsilon^4 |\varphi_1|_6^6 + C\varepsilon |\varphi_0|_\infty^2 y_\varepsilon^2 + C\varepsilon^2 (1 + |\varphi_0|_\infty^4) y_\varepsilon^3 + C\varepsilon y_\varepsilon z_\varepsilon. \tag{4.22}
$$

The fifth term is bounded easily by

$$
|I_5| \leq |\partial_z \varphi_0|_\infty |v_\varepsilon|_2 |R_\varepsilon|_2 \leq |\partial_z \varphi_0|_\infty y_\varepsilon. \tag{4.23}
$$

Using the inequalities

$$
|\partial_z R_\varepsilon|_2 \leq |R_\varepsilon|_2^{\frac{1}{2}} |\partial_z^2 R_\varepsilon|_2^{\frac{1}{2}},
$$

$$
|\partial_x R_\varepsilon|_2 \leq |R_\varepsilon|_2^{\frac{1}{2}} |\partial_x^2 R_\varepsilon|_2^{\frac{1}{2}},
$$

$$
|\partial_x \partial_z R_\varepsilon|_2 \leq |\partial_x^2 R_\varepsilon|_2^{\frac{1}{2}} |\partial_z^2 R_\varepsilon|_2^{\frac{1}{2}},
$$

obtained by integration by parts, and the anisotropic Sobolev inequality (3.11) with $p = 4$, one has the following estimate:

$$
\begin{aligned}
|\partial_x R_\varepsilon|_4 &\le C|\partial_x R_\varepsilon|_2^{\frac12}(|\partial_x R_\varepsilon|_2 + |\partial_x^2 R_\varepsilon|_2)^{\frac14}(|\partial_x R_\varepsilon|_2 + |\partial_x\partial_z R_\varepsilon|_2)^{\frac14}\\
&\le C|R_\varepsilon|_2^{\frac14}|\partial_x^2 R_\varepsilon|_2^{\frac14}(|R_\varepsilon|_2 + |\partial_x^2 R_\varepsilon|_2)^{\frac14}(|R_\varepsilon|_2^{\frac12}|\partial_x^2 R_\varepsilon|_2^{\frac12} + |\partial_z^2 R_\varepsilon|_2^{\frac12}|\partial_x^2 R_\varepsilon|_2^{\frac12})^{\frac14}\\
&\le C|R_\varepsilon|_2^{\frac14}|\partial_x^2 R_\varepsilon|_2^{\frac38}(|R_\varepsilon|_2 + |\partial_x^2 R_\varepsilon|_2)^{\frac14}(|R_\varepsilon|_2 + |\partial_z^2 R_\varepsilon|_2)^{\frac18},
\end{aligned}
$$

which implies, using inequality (4.15), the estimate

$$
\begin{aligned}
\varepsilon^{\frac54}|\partial_x R_\varepsilon|_4 &\le C|R_\varepsilon|_2^{\frac14}(\varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac38}(\varepsilon^2|R_\varepsilon|_2 + \varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac14}(|R_\varepsilon|_2 + |\partial_z^2 R_\varepsilon|_2)^{\frac18}\\
&\le C y_\varepsilon^{\frac18} z_\varepsilon^{\frac{5}{16}}(y_\varepsilon^{\frac{1}{16}} + z_\varepsilon^{\frac{1}{16}})\\
&\le C y_\varepsilon^{\frac{3}{16}} z_\varepsilon^{\frac{5}{16}} + C y_\varepsilon^{\frac18} z_\varepsilon^{\frac38}.
\end{aligned}
\tag{4.24}
$$

In a very similar way, it can be shown that

$$
\varepsilon^{\frac14}|\partial_z R_\varepsilon|_4 \le C y_\varepsilon^{\frac{3}{16}} z_\varepsilon^{\frac{5}{16}} + C y_\varepsilon^{\frac18} z_\varepsilon^{\frac38}. \tag{4.25}
$$

Using the Sobolev inequality (3.11) with $p = 4$, we obtain

$$
\begin{aligned}
\varepsilon^{\frac14}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4 &\le C\left|\frac{u_\varepsilon}{\varepsilon}\right|_2^{\frac12}\left(\varepsilon\left|\frac{u_\varepsilon}{\varepsilon}\right|_2 + \varepsilon\left|\partial_x\frac{u_\varepsilon}{\varepsilon}\right|_2\right)^{\frac14}\left(\left|\frac{u_\varepsilon}{\varepsilon}\right|_2 + \left|\partial_z\frac{u_\varepsilon}{\varepsilon}\right|_2\right)^{\frac14}\\
&\le C y_\varepsilon^{\frac14} z_\varepsilon^{\frac14},
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
\varepsilon^{\frac14}|v_\varepsilon|_4 &\le C|v_\varepsilon|_2^{\frac12}(\varepsilon|v_\varepsilon|_2 + \varepsilon|\partial_x v_\varepsilon|_2)^{\frac14}(|v_\varepsilon|_2 + |\partial_z v_\varepsilon|_2)^{\frac14}\\
&\le C y_\varepsilon^{\frac14} z_\varepsilon^{\frac14}.
\end{aligned}
\tag{4.27}
$$

Let us remark that we used here the fact that u_ε and v_ε vanish on the boundaries $\{z = \pm 1\}$, and so that we can use the Poincaré inéquality (in the z direction)

$$
|u_\varepsilon|_2 \le C|\partial_z u_\varepsilon|_2, \text{ and } |v_\varepsilon|_2 \le C|\partial_z v_\varepsilon|_2.
$$

Then, thanks to (4.24)-(4.27), the last term is bounded by

$$
\begin{aligned}
|I_6| &\le \varepsilon^2|\varphi_1|_2\left|\frac{u_\varepsilon}{\varepsilon}\right|_4|\partial_x R_\varepsilon|_4 + \varepsilon|\varphi_1|_2|v_\varepsilon|_4|\partial_z R_\varepsilon|_4\\
&\le \varepsilon^{\frac12}|\varphi_1|_2\left(\varepsilon^{\frac14}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)(\varepsilon^{\frac54}|\partial_x R_\varepsilon|_4) + \varepsilon^{\frac12}|\varphi_1|_2(\varepsilon^{\frac14}|v_\varepsilon|_4)(\varepsilon^{\frac14}|\partial_z R_\varepsilon|_4)\\
&\le C\varepsilon^{\frac12}|\varphi_1|_2(y_\varepsilon^{\frac{7}{16}} z_\varepsilon^{\frac{9}{16}} + y_\varepsilon^{\frac38} z_\varepsilon^{\frac58})\\
&\le \frac{1}{50} z_\varepsilon + C\varepsilon^{\frac87}|\varphi_1|_2^{\frac{16}{7}} y_\varepsilon + C\varepsilon^{\frac43}|\varphi_1|_2^{\frac83} y_\varepsilon.
\end{aligned}
\tag{4.28}
$$

• ***Step 2 - Terms issued from the L^2 estimate on W_ε***

The first two terms are easily controlled in the following way:

$$
\begin{aligned}
|J_1| &\le \alpha^2\varepsilon|\partial_z^2\varphi_0|_\infty|\partial_x\varphi_1|_2\left|\frac{u_\varepsilon}{\varepsilon}\right|_2 + \alpha^2\varepsilon|\partial_z^2\varphi_0|_\infty|\partial_x R_\varepsilon|_2\left|\frac{u_\varepsilon}{\varepsilon}\right|_2\\
&\le \alpha^2\varepsilon|\partial_z^2\varphi_0|_\infty|\partial_x\varphi_1|_2 y_\varepsilon^{\frac12} + \alpha^2|\partial_z^2\varphi_0|_\infty|R_\varepsilon|_2^{\frac12}(\varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac12}\left|\frac{u_\varepsilon}{\varepsilon}\right|_2\\
&\le \frac{1}{50} z_\varepsilon + C|\partial_z^2\varphi_0|_\infty^{\frac43} y_\varepsilon + C\varepsilon|\partial_z^2\varphi_0|_\infty|\partial_x\varphi_1|_2 y_\varepsilon^{\frac12},
\end{aligned}
\tag{4.29}
$$

and

$$\begin{aligned}|J_2| &\le \alpha^2|\partial_z^2\varphi_0|_\infty|\partial_z\varphi_1|_2|v_\varepsilon|_2+\alpha^2|\partial_z^2\varphi_0|_\infty|\partial_z R_\varepsilon|_2|v_\varepsilon|_2\\ &\le \alpha^2|\partial_z^2\varphi_0|_\infty|\partial_z\varphi_1|_2 y_\varepsilon^{\frac12}+\alpha^2|\partial_z^2\varphi_0|_\infty|R_\varepsilon|_2^{\frac12}|\partial_z^2R_\varepsilon|_2^{\frac12}|v_\varepsilon|_2\\ &\le \frac{1}{50}z_\varepsilon+C|\partial_z^2\varphi_0|_\infty|\partial_z\varphi_1|_2y_\varepsilon^{\frac12}+C|\partial_z^2\varphi_0|_\infty^{\frac43}y_\varepsilon.\end{aligned}\tag{4.30}$$

We need now a new estimate of $|\partial_x R_\varepsilon|_4$ which requires a higher power of ε, but which gives a control with a lower power of z_ε. This is absolutely necessary in view of the control of these terms by the coercive terms (that is to say by z_ε). More precisely, thanks to the Sobolev inequality (3.11), it follows that

$$\begin{aligned}\varepsilon^{\frac32}|\partial_xR_\varepsilon|_4 &\le C\varepsilon^{\frac32}|\partial_xR_\varepsilon|_2^{\frac12}(|\partial_xR_\varepsilon|_2+|\partial_x^2R_\varepsilon|_2)^{\frac14}(|\partial_xR_\varepsilon|_2+|\partial_x\partial_zR_\varepsilon|_2)^{\frac14}\\ &\le C(\varepsilon^{\frac32}|\partial_xR_\varepsilon|_2)^{\frac12}(\varepsilon^2|\partial_xR_\varepsilon|_2+\varepsilon^2|\partial_x^2R_\varepsilon|_2)^{\frac14}\left(|R_\varepsilon|_2^{\frac12}(\varepsilon^2|\partial_x^2R_\varepsilon|_2)^{\frac12}+|\partial_z^2R_\varepsilon|_2^{\frac12}(\varepsilon^2|\partial_x^2R_\varepsilon|_2)^{\frac12}\right)^{\frac14}\\ &\le Cy_\varepsilon^{\frac14}(y_\varepsilon^{\frac12}+z_\varepsilon^{\frac12})^{\frac14}(y_\varepsilon^{\frac14}+z_\varepsilon^{\frac14})^{\frac14}z_\varepsilon^{\frac1{16}}\\ &\le Cy_\varepsilon^{\frac14}(y_\varepsilon^{\frac3{16}}+z_\varepsilon^{\frac3{16}})z_\varepsilon^{\frac1{16}}\le Cy_\varepsilon^{\frac7{16}}z_\varepsilon^{\frac1{16}}+Cy_\varepsilon^{\frac14}z_\varepsilon^{\frac14}.\end{aligned}$$

Hence, the third term is estimated using (4.26) and the previous inequality, by

$$\begin{aligned}|J_3| &\le \alpha^2\varepsilon^{\frac74}|\partial_z^2\varphi_1|_2|\partial_x\varphi_1|_4\left(\varepsilon^{\frac14}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)+\alpha^2\varepsilon^{\frac14}|\partial_z^2\varphi_1|_2(\varepsilon^{\frac32}|\partial_xR_\varepsilon|_4)\left(\varepsilon^{\frac14}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)\\ &\quad+\alpha^2\varepsilon^{\frac74}|\partial_z^2R_\varepsilon|_2|\partial_x\varphi_1|_4\left(\varepsilon^{\frac14}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)+\alpha^2\varepsilon^{\frac14}|\partial_z^2R_\varepsilon|_2(\varepsilon^{\frac32}|\partial_xR_\varepsilon|_4)\left(\varepsilon^{\frac14}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)\\ &\le C\varepsilon^{\frac74}|\partial_z^2\varphi_1|_2|\partial_x\varphi_1|_4y_\varepsilon^{\frac14}z_\varepsilon^{\frac14}+C\varepsilon^{\frac14}|\partial_z^2\varphi_1|_2(y_\varepsilon^{\frac7{16}}z_\varepsilon^{\frac1{16}}+y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})y_\varepsilon^{\frac14}z_\varepsilon^{\frac14}\\ &\quad+C\varepsilon^{\frac74}|\partial_x\varphi_1|_4y_\varepsilon^{\frac14}z_\varepsilon^{\frac34}+C\varepsilon^{\frac14}(y_\varepsilon^{\frac7{16}}z_\varepsilon^{\frac1{16}}+y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})y_\varepsilon^{\frac14}z_\varepsilon^{\frac34}\\ &\le \frac1{50}z_\varepsilon+C\varepsilon^{\frac73}|\partial_z^2\varphi_1|_2^{\frac43}|\partial_x\varphi_1|_4^{\frac43}y_\varepsilon^{\frac13}+C\varepsilon^{\frac4{11}}|\partial_z^2\varphi_1|_2^{\frac{16}{11}}y_\varepsilon+C\varepsilon^{\frac12}|\partial_z^2\varphi_1|_2^2y_\varepsilon\\ &\quad+C\varepsilon^7|\partial_x\varphi_1|_4^4y_\varepsilon+C\varepsilon^{\frac43}y_\varepsilon^{\frac{11}3}+C\varepsilon^{\frac14}y_\varepsilon^{\frac12}z_\varepsilon.\end{aligned}\tag{4.31}$$

The fourth term is bounded in a similar way, replacing formally $\partial_z^2\varphi_1$ by $\varepsilon^2\partial_x^2\varphi_1$ and $\partial_z^2R_\varepsilon$ by $\varepsilon^2\partial_x^2R_\varepsilon$:

$$\begin{aligned}|J_4| &\le \frac1{50}z_\varepsilon+C\varepsilon^{\frac{15}3}|\partial_x^2\varphi_1|_2^{\frac43}|\partial_x\varphi_1|_4^{\frac43}y_\varepsilon^{\frac13}+C\varepsilon^{\frac{36}{11}}|\partial_x^2\varphi_1|_2^{\frac{16}{11}}y_\varepsilon+C\varepsilon^{\frac92}|\partial_x^2\varphi_1|_2^2y_\varepsilon\\ &\quad+C\varepsilon^7|\partial_x\varphi_1|_4^4y_\varepsilon+C\varepsilon^{\frac43}y_\varepsilon^{\frac{11}3}+C\varepsilon^{\frac14}y_\varepsilon^{\frac12}z_\varepsilon.\end{aligned}\tag{4.32}$$

As far as the fifth and sixth terms are concerned, we have

$$\begin{aligned}|J_5| &\le \alpha^2\varepsilon^2|\partial_z\varphi_0|_\infty|\partial_x^2\varphi_1|_2|v_\varepsilon|_2+\alpha^2\varepsilon^2|\partial_z\varphi_0|_\infty|\partial_x^2R_\varepsilon|_2|v_\varepsilon|_2\\ &\le \alpha^2\varepsilon^2|\partial_z\varphi_0|_\infty|\partial_x^2\varphi_1|_2y_\varepsilon^{\frac12}+\alpha^2|\partial_z\varphi_0|_\infty y_\varepsilon^{\frac12}z_\varepsilon^{\frac12}\\ &\le \frac1{50}z_\varepsilon+C\varepsilon^2|\partial_z\varphi_0|_\infty|\partial_x^2\varphi_1|_2y_\varepsilon^{\frac12}+C|\partial_z\varphi_0|_\infty^2y_\varepsilon,\end{aligned}\tag{4.33}$$

and

$$\begin{aligned}|J_6| &\le \alpha^2|\partial_z\varphi_0|_\infty|\partial_z^2\varphi_1|_2|v_\varepsilon|_2+\alpha^2|\partial_z\varphi_0|_\infty|\partial_z^2R_\varepsilon|_2|v_\varepsilon|_2\\ &\le \alpha^2|\partial_z\varphi_0|_\infty|\partial_z^2\varphi_1|_2y_\varepsilon^{\frac12}+\alpha^2|\partial_z\varphi_0|_\infty y_\varepsilon^{\frac12}z_\varepsilon^{\frac12}\\ &\le \frac1{50}z_\varepsilon+C|\partial_z\varphi_0|_\infty|\partial_z^2\varphi_1|_2y_\varepsilon^{\frac12}+C|\partial_z\varphi_0|_\infty^2y_\varepsilon.\end{aligned}\tag{4.34}$$

Finally, the last two terms are estimated exactly as the third and the fourth ones. We first have to prove a new estimate on $|\partial_z R_\varepsilon|_4$, thanks to (3.11)

$$\begin{aligned}\varepsilon^{\frac12}|\partial_z R_\varepsilon|_4 &\le C\varepsilon^{\frac12}|\partial_z R_\varepsilon|_2^{\frac12}(|\partial_z R_\varepsilon|_2+|\partial_x\partial_z R_\varepsilon|_2)^{\frac14}(|\partial_z R_\varepsilon|_2+|\partial_z^2 R_\varepsilon|_2)^{\frac14}\\ &\le C(\varepsilon^{\frac12}|\partial_z R_\varepsilon|_2)^{\frac12}\left(\varepsilon|R_\varepsilon|_2^{\frac12}|\partial_z^2 R_\varepsilon|_2^{\frac12}+(\varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac12}|\partial_z^2 R_\varepsilon|_2^{\frac12}\right)^{\frac14}(|R_\varepsilon|_2+|\partial_z^2 R_\varepsilon|_2)^{\frac14}\\ &\le Cy_\varepsilon^{\frac14}z_\varepsilon^{\frac1{16}}(y_\varepsilon^{\frac3{16}}+z_\varepsilon^{\frac3{16}})\\ &\le Cy_\varepsilon^{\frac7{16}}z_\varepsilon^{\frac1{16}}+Cy_\varepsilon^{\frac14}z_\varepsilon^{\frac14}.\end{aligned}\tag{4.35}$$

Therefore, with (4.27) and the above, we find

$$\begin{aligned}|J_7| &\le \alpha^2\varepsilon|\partial_z^2\varphi_1|_2|\partial_z\varphi_1|_4|v_\varepsilon|_4+\alpha^2\varepsilon|\partial_z^2\varphi_1|_2|\partial_z R_\varepsilon|_4|v_\varepsilon|_4\\ &\qquad+\alpha^2\varepsilon|\partial_z^2 R_\varepsilon|_2|\partial_z\varphi_1|_4|v_\varepsilon|_4+\alpha^2\varepsilon|\partial_z^2 R_\varepsilon|_2|\partial_z R_\varepsilon|_4|v_\varepsilon|_4\\ &\le C\varepsilon^{\frac34}|\partial_z^2\varphi_1|_2^2|\partial_z\varphi_1|_4 y_\varepsilon^{\frac14}z_\varepsilon^{\frac14}+C\varepsilon^{\frac14}|\partial_z^2\varphi_1|_2(y_\varepsilon^{\frac7{16}}z_\varepsilon^{\frac1{16}}+y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})y_\varepsilon^{\frac14}z_\varepsilon^{\frac14}\\ &\qquad+C\varepsilon^{\frac34}|\partial_z\varphi_1|_4 y_\varepsilon^{\frac14}z_\varepsilon^{\frac34}+C\varepsilon^{\frac14}z_\varepsilon^{\frac12}(y_\varepsilon^{\frac7{16}}z_\varepsilon^{\frac1{16}}+y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})y_\varepsilon^{\frac14}z_\varepsilon^{\frac14}\\ &\le \frac1{50}z_\varepsilon+C\varepsilon|\partial_z^2\varphi_1|_2^{\frac43}|\partial_z\varphi_1|_4^{\frac43}y_\varepsilon^{\frac13}+C\varepsilon^{\frac4{11}}|\partial_z^2\varphi_1|_2^{\frac{16}{11}}y_\varepsilon+C\varepsilon^{\frac12}|\partial_z^2\varphi_1|_2^2 y_\varepsilon\\ &\qquad+C\varepsilon^3|\partial_z\varphi_1|_4^4 y_\varepsilon+C\varepsilon^{\frac43}y_\varepsilon^{\frac{11}3}+C\varepsilon^{\frac14}y_\varepsilon^{\frac12}z_\varepsilon,\end{aligned}\tag{4.36}$$

and, replacing $\partial_z^2 R_\varepsilon$ by $\varepsilon^2\partial_x^2 R_\varepsilon$ and $\partial_z^2\varphi_1$ by $\varepsilon^2\partial_x^2\varphi_1$, one has

$$\begin{aligned}|J_8| &\le \frac1{50}z_\varepsilon+C\varepsilon^{\frac{11}3}|\partial_x^2\varphi_1|_2^{\frac43}|\partial_z\varphi_1|_4^{\frac43}y_\varepsilon^{\frac13}+C\varepsilon^{\frac{36}{11}}|\partial_x^2\varphi_1|_2^{\frac{16}{11}}y_\varepsilon+C\varepsilon^{\frac92}|\partial_x^2\varphi_1|_2^2 y_\varepsilon\\ &\qquad+C\varepsilon^3|\partial_z\varphi_1|_4^4 y_\varepsilon+C\varepsilon^{\frac43}y_\varepsilon^{\frac{11}3}+C\varepsilon^{\frac14}y_\varepsilon^{\frac12}z_\varepsilon.\end{aligned}\tag{4.37}$$

We just have seen that the bounds obtained on these last terms absolutely require to have in the energy y_ε considered here the terms $\varepsilon^3|\partial_x R_\varepsilon|_2$ and $\varepsilon|\partial_z R_\varepsilon|_2$ issued from the H^1 estimates on R_ε.

• ***Step 3 - Terms issued from the estimate on $\varepsilon^3|\partial_x R_\varepsilon|_2^2$***

The first two terms do not present any particular difficulty:

$$|K_1|\le\varepsilon^5|F''(\varphi_0)|_\infty|\partial_x^2 R_\varepsilon|_2^2\le\varepsilon|F''(\varphi_0)|_\infty z_\varepsilon,\tag{4.38}$$

and

$$\begin{aligned}|K_2| &\le \varepsilon^3|F''(\varphi_0)|_\infty|\partial_x^2 R_\varepsilon|_2|\partial_z^2 R_\varepsilon|_2+\varepsilon^3|\partial_z F''(\varphi_0)|_\infty|\partial_z R_\varepsilon|_2|\partial_x^2 R_\varepsilon|_2+\varepsilon^3|\partial_z^2 F''(\varphi_0)|_\infty|R_\varepsilon|_2|\partial_x^2 R_\varepsilon|_2\\ &\le C\varepsilon|F''(\varphi_0)|_\infty z_\varepsilon+C\varepsilon^{\frac12}|\partial_z F''(\varphi_0)|_\infty y_\varepsilon^{\frac12}z_\varepsilon^{\frac12}+\varepsilon|\partial_z^2 F''(\varphi_0)|_\infty y_\varepsilon^{\frac12}z_\varepsilon^{\frac12}\\ &\le \frac1{50}z_\varepsilon+C\varepsilon|F''(\varphi_0)|_\infty z_\varepsilon+C\varepsilon|\partial_z F''(\varphi_0)|_\infty^2 y_\varepsilon+C\varepsilon^2|\partial_z^2 F''(\varphi_0)|_\infty^2 y_\varepsilon.\end{aligned}\tag{4.39}$$

The third one does not require additional work, as such a term has previously been treated (with a small power of ε) in (4.32) and (4.37). Hence, we have here

$$\begin{aligned}|K_3| &= \varepsilon^4\left|\int_{\Omega_0}W_\varepsilon.\nabla\varphi_1\partial_x^2 R_\varepsilon\right|+\varepsilon^4\left|\int_{\Omega_0}W_\varepsilon.\nabla R_\varepsilon\partial_x^2 R_\varepsilon\right|\\ &\le \frac1{50}z_\varepsilon+C\varepsilon^{\frac{20}3}y_\varepsilon^{\frac{11}3}+C\varepsilon^{\frac54}y_\varepsilon^{\frac12}z_\varepsilon+C\varepsilon^{11}|\partial_x\varphi_1|_4^4 y_\varepsilon+C\varepsilon^7|\partial_z\varphi_1|_4^4 y_\varepsilon.\end{aligned}\tag{4.40}$$

The fourth term can be controlled by

$$|K_4|\le\varepsilon^3|\partial_z\varphi_0|_\infty|v_\varepsilon|_2|\partial_x^2 R_\varepsilon|_2\le\varepsilon|\partial_z\varphi_0|_\infty y_\varepsilon^{\frac12}z_\varepsilon^{\frac12}\le\frac1{50}z_\varepsilon+C\varepsilon^2|\partial_z\varphi_0|_\infty^2 y_\varepsilon.\tag{4.41}$$

In order to estimate the next two terms, we have to find a bound on $|R_\varepsilon|_8$ with the anisotropic Sobolev inequality (3.13) for $p=8$ and to use the coercive terms issued from the estimates on $\varepsilon^3|\partial_x R_\varepsilon|_2^2$ and on $\varepsilon|\partial_z R_\varepsilon|_2^2$. We obtain

$$
\begin{aligned}
\varepsilon^{\frac{9}{16}}|R_\varepsilon|_8 &\le C\varepsilon^{\frac{9}{16}}|R_\varepsilon|_2^{\frac{5}{8}}(|R_\varepsilon|_2+|\partial_x^2 R_\varepsilon|_2)^{\frac{3}{16}}(|R_\varepsilon|_2+|\partial_z^2 R_\varepsilon|_2)^{\frac{3}{16}} \\
&\le C|R_\varepsilon|_2^{\frac{5}{8}}\left(\varepsilon^{\frac{10}{4}}|R_\varepsilon|_2+(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_2)^{\frac{1}{2}}(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2)^{\frac{1}{2}}\right)^{\frac{3}{16}} \\
&\qquad \times\left(\varepsilon^{\frac{1}{2}}|R_\varepsilon|_2+(\varepsilon^{\frac{1}{2}}|\partial_z R_\varepsilon|_2)^{\frac{1}{2}}(\varepsilon^{\frac{1}{2}}|\partial_z^3 R_\varepsilon|_2)^{\frac{1}{2}}\right)^{\frac{3}{16}} \qquad (4.42)\\
&\le C y_\varepsilon^{\frac{5}{16}}\left(y_\varepsilon^{\frac{1}{2}}+y_\varepsilon^{\frac{1}{4}}z_\varepsilon^{\frac{1}{4}}\right)^{\frac{3}{8}} \\
&\le C y_\varepsilon^{\frac{1}{2}}+C y_\varepsilon^{\frac{13}{32}}z_\varepsilon^{\frac{3}{32}}.
\end{aligned}
$$

We can now estimate the fifth term of $\varepsilon^3\times$(4.11) as follows:

$$
\begin{aligned}
|K_5| \le\; & C\varepsilon^6|\varphi_0|_\infty|\varphi_1|_4|\partial_x\varphi_1|_4|\partial_x^3 R_\varepsilon|_2+C\varepsilon^6|\varphi_0|_\infty|R_\varepsilon|_4|\partial_x\varphi_1|_4|\partial_x^3 R_\varepsilon|_2 \\
&+C\varepsilon^6|\varphi_0|_\infty|\varphi_1|_4|\partial_x R_\varepsilon|_4|\partial_x^3 R_\varepsilon|_2+C\varepsilon^6|\varphi_0|_\infty|R_\varepsilon|_4|\partial_x R_\varepsilon|_4|\partial_x^3 R_\varepsilon|_2 \\
&+C\varepsilon^7|\varphi_1|_8^2|\partial_x\varphi_1|_4|\partial_x^3 R_\varepsilon|_2+C\varepsilon^7|\varphi_1|_8^2|\partial_x R_\varepsilon|_4|\partial_x^3 R_\varepsilon|_2 \\
&+C\varepsilon^7|R_\varepsilon|_8^2|\partial_x\varphi_1|_4|\partial_x^3 R_\varepsilon|_2+C\varepsilon^7|R_\varepsilon|_8^2|\partial_x R_\varepsilon|_4|\partial_x^3 R_\varepsilon|_2
\end{aligned}
$$

$$
\begin{aligned}
\le\; & C\varepsilon^{\frac{5}{2}}|\varphi_0|_\infty|\varphi_1|_4|\partial_x\varphi_1|_4(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2)+C\varepsilon^{\frac{9}{4}}|\varphi_0|_\infty(\varepsilon^{\frac{1}{4}}|R_\varepsilon|_4)|\partial_x\varphi_1|_4(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2) \\
&+C\varepsilon|\varphi_0|_\infty|\varphi_1|_4(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2)+C\varepsilon^{\frac{3}{4}}|\varphi_0|_\infty(\varepsilon^{\frac{1}{4}}|R_\varepsilon|_4)(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2) \\
&+C\varepsilon^{\frac{7}{2}}|\varphi_1|_8^2|\partial_x\varphi_1|_4(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2)+C\varepsilon^2|\varphi_1|_8^2(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2) \\
&+C\varepsilon^{\frac{19}{8}}(\varepsilon^{\frac{9}{16}}|R_\varepsilon|_8)^2|\partial_x\varphi_1|_4(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2)+C\varepsilon^{\frac{7}{8}}(\varepsilon^{\frac{9}{16}}|R_\varepsilon|_8)^2(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2),
\end{aligned}
$$

which gives, using (4.19) and (4.42), the following bound:

$$
\begin{aligned}
|K_5| \le\; & C\varepsilon^{\frac{5}{2}}|\varphi_0|_\infty|\varphi_1|_4|\partial_x\varphi_1|_4 z_\varepsilon^{\frac{1}{2}}+C\varepsilon^{\frac{9}{4}}|\varphi_0|_\infty|\partial_x\varphi_1|_4(y_\varepsilon^{\frac{1}{2}}+y_\varepsilon^{\frac{3}{8}}z_\varepsilon^{\frac{1}{8}})z_\varepsilon^{\frac{1}{2}} \\
&+C\varepsilon|\varphi_0|_\infty|\varphi_1|_4(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}}+y_\varepsilon^{\frac{1}{4}}z_\varepsilon^{\frac{1}{4}})z_\varepsilon^{\frac{1}{2}}+C\varepsilon^{\frac{3}{4}}|\varphi_0|_\infty(y_\varepsilon^{\frac{1}{2}}+y_\varepsilon^{\frac{3}{8}}z_\varepsilon^{\frac{1}{8}})(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}}+y_\varepsilon^{\frac{1}{4}}z_\varepsilon^{\frac{1}{4}})z_\varepsilon^{\frac{1}{2}} \\
&+C\varepsilon^{\frac{7}{2}}|\varphi_1|_8^2|\partial_x\varphi_1|_4 z_\varepsilon^{\frac{1}{2}}+C\varepsilon^2|\varphi_1|_8^2(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}}+y_\varepsilon^{\frac{1}{4}}z_\varepsilon^{\frac{1}{4}})z_\varepsilon^{\frac{1}{2}} \\
&+C\varepsilon^{\frac{19}{8}}|\partial_x\varphi_1|_4\left(y_\varepsilon^{\frac{1}{2}}+y_\varepsilon^{\frac{13}{32}}z_\varepsilon^{\frac{3}{32}}\right)^2 z_\varepsilon^{\frac{1}{2}}+C\varepsilon^{\frac{7}{8}}\left(y_\varepsilon^{\frac{1}{2}}+y_\varepsilon^{\frac{13}{32}}z_\varepsilon^{\frac{3}{32}}\right)^2(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}}+y_\varepsilon^{\frac{1}{4}}z_\varepsilon^{\frac{1}{4}})z_\varepsilon^{\frac{1}{2}}
\end{aligned}
\qquad (4.43)
$$

$$
\begin{aligned}
\le\; & \frac{1}{50}z_\varepsilon+C\varepsilon^5|\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_x\varphi_1|_4^2+C\varepsilon^{\frac{9}{2}}|\varphi_0|_\infty^2|\partial_x\varphi_1|_4^2 y_\varepsilon+C\varepsilon^6|\varphi_0|_\infty^{\frac{8}{3}}|\partial_x\varphi_1|_4^{\frac{8}{3}}y_\varepsilon \\
&+C\varepsilon^{\frac{16}{7}}|\varphi_0|_\infty^{\frac{16}{7}}|\varphi_1|_4^{\frac{16}{7}}y_\varepsilon+C\varepsilon^4|\varphi_0|_\infty^4|\varphi_1|_4^4 y_\varepsilon+C\varepsilon^{\frac{12}{7}}|\varphi_0|_\infty^{\frac{16}{7}}y_\varepsilon^{\frac{15}{7}}+C\varepsilon^6|\varphi_0|_\infty^8 y_\varepsilon^5 \\
&+C\varepsilon^7|\varphi_1|_8^4|\partial_x\varphi_1|_4^2+C\varepsilon^{\frac{32}{7}}|\varphi_1|_8^{\frac{32}{7}}y_\varepsilon+C\varepsilon^8|\varphi_1|_8^8 y_\varepsilon \\
&+C\varepsilon^{\frac{19}{4}}|\partial_x\varphi_1|_4^2 y_\varepsilon^2+C\varepsilon^{\frac{38}{5}}|\partial_x\varphi_1|_4^{\frac{16}{5}}y_\varepsilon^{\frac{13}{5}}+C\varepsilon^2 y_\varepsilon^{\frac{23}{7}}+C\varepsilon^{14}y_\varepsilon^{17}.
\end{aligned}
$$

The sixth term is very similar to the previous one, except that $\partial_z\varphi_0$ does not vanish in general, so that we have

$$
\begin{aligned}
|K_6| \le\ & C\varepsilon^4|\partial_z\varphi_0|_\infty|\varphi_1|_4^2|\partial_z\partial_x^2 R_\varepsilon|_2 + C\varepsilon^4|\partial_z\varphi_0|_\infty|R_\varepsilon|_4^2|\partial_z\partial_x^2 R_\varepsilon|_2\\
&+ C\varepsilon^4|\varphi_0|_\infty|\varphi_1|_4|\partial_z\varphi_1|_4|\partial_x\partial_z^2 R_\varepsilon|_2 + C\varepsilon^4|\varphi_0|_\infty|R_\varepsilon|_4|\partial_z\varphi_1|_4|\partial_z\partial_x^2 R_\varepsilon|_2\\
&+ C\varepsilon^4|\varphi_0|_\infty|\varphi_1|_4|\partial_z R_\varepsilon|_4|\partial_z\partial_x^2 R_\varepsilon|_2 + C\varepsilon^4|\varphi_0|_\infty|R_\varepsilon|_4|\partial_z R_\varepsilon|_4|\partial_z\partial_x^2 R_\varepsilon|_2\\
&+ C\varepsilon^5|\varphi_1|_8^2|\partial_z\varphi_1|_4|\partial_z\partial_x^2 R_\varepsilon|_2 + C\varepsilon^5|\varphi_1|_8^2|\partial_z R_\varepsilon|_4|\partial_z\partial_x^2 R_\varepsilon|_2\\
&+ C\varepsilon^5|R_\varepsilon|_8^2|\partial_z\varphi_1|_4|\partial_z\partial_x^2 R_\varepsilon|_2 + C\varepsilon^5|R_\varepsilon|_8^2|\partial_z R_\varepsilon|_4|\partial_z\partial_x^2 R_\varepsilon|_2
\end{aligned}
$$

$$
\begin{aligned}
\le\ & C\varepsilon^{\frac32}|\partial_z\varphi_0|_\infty|\varphi_1|_4^2(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2) + C\varepsilon|\partial_z\varphi_0|_\infty(\varepsilon^{\frac14}|R_\varepsilon|_4)^2(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2)\\
&+ C\varepsilon^{\frac32}|\varphi_0|_\infty|\varphi_1|_4|\partial_z\varphi_1|_4(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2) + C\varepsilon^{\frac54}|\varphi_0|_\infty(\varepsilon^{\frac14}|R_\varepsilon|_4)|\partial_z\varphi_1|_4(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2)\\
&+ C\varepsilon^4|\varphi_0|_\infty|\varphi_1|_4(\varepsilon^{\frac12}|\partial_z R_\varepsilon|_4)(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2) + C\varepsilon^{\frac34}|\varphi_0|_\infty(\varepsilon^{\frac14}|R_\varepsilon|_4)(\varepsilon^{\frac12}|\partial_z R_\varepsilon|_4)(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2)\\
&+ C\varepsilon^{\frac52}|\varphi_1|_8^2|\partial_z\varphi_1|_4(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2) + C\varepsilon^2|\varphi_1|_8^2(\varepsilon^{\frac12}|\partial_z R_\varepsilon|_4)(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2)\\
&+ C\varepsilon^{\frac{11}{8}}(\varepsilon^{\frac{9}{16}}|R_\varepsilon|_8)^2|\partial_z\varphi_1|_4(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2) + C\varepsilon^{\frac78}(\varepsilon^{\frac{9}{16}}|R_\varepsilon|_8)^2(\varepsilon^{\frac12}|\partial_z R_\varepsilon|_4)(\varepsilon^{\frac52}|\partial_z\partial_x^2 R_\varepsilon|_2),
\end{aligned}
$$

and then, using (4.19) and (4.42), one obtains

$$
\begin{aligned}
|K_6| \le\ & C\varepsilon^{\frac32}|\partial_z\varphi_0|_\infty|\varphi_1|_4^2 z_\varepsilon^{\frac12} + C\varepsilon|\partial_z\varphi_0|_\infty(y_\varepsilon^{\frac12} + y_\varepsilon^{\frac38}z_\varepsilon^{\frac18})z_\varepsilon^{\frac12}\\
&+ C\varepsilon^{\frac32}|\varphi_0|_\infty|\varphi_1|_4|\partial_z\varphi_1|_4 z_\varepsilon^{\frac12} + C\varepsilon^{\frac54}|\varphi_0|_\infty|\partial_z\varphi_1|_4(y_\varepsilon^{\frac12} + y_\varepsilon^{\frac38}z_\varepsilon^{\frac18})z_\varepsilon^{\frac12}\\
&+ C\varepsilon|\varphi_0|_\infty|\varphi_1|_4(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}} + y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})z_\varepsilon^{\frac12} + C\varepsilon^{\frac34}|\varphi_0|_\infty(y_\varepsilon^{\frac12} + y_\varepsilon^{\frac38}z_\varepsilon^{\frac18})(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}} + y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})z_\varepsilon^{\frac12}\\
&+ C\varepsilon^{\frac52}|\varphi_1|_8^2|\partial_z\varphi_1|_4 z_\varepsilon^{\frac12} + C\varepsilon^2|\varphi_1|_8^2(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}} + y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})z_\varepsilon^{\frac12}\\
&+ C\varepsilon^{\frac{11}{8}}|\partial_z\varphi_1|_4\left(y_\varepsilon^{\frac12} + y_\varepsilon^{\frac{13}{32}}z_\varepsilon^{\frac{3}{32}}\right)^2 z_\varepsilon^{\frac12} + C\varepsilon^{\frac78}\left(y_\varepsilon^{\frac12} + y_\varepsilon^{\frac{13}{32}}z_\varepsilon^{\frac{3}{32}}\right)^2(y_\varepsilon^{\frac{7}{16}}z_\varepsilon^{\frac{1}{16}} + y_\varepsilon^{\frac14}z_\varepsilon^{\frac14})z_\varepsilon^{\frac12}
\end{aligned}
\tag{4.44}
$$

$$
\begin{aligned}
\le\ & \frac{1}{50}z_\varepsilon + C\varepsilon^3|\partial_z\varphi_0|_\infty^2|\varphi_1|_4^4 + C\varepsilon^2|\partial_z\varphi_0|_\infty^2 y_\varepsilon + C\varepsilon^{\frac83}|\partial_z\varphi_0|_\infty^{\frac83} y_\varepsilon\\
&+ C\varepsilon^3|\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_z\varphi_1|_4^2 + C\varepsilon^{\frac52}|\varphi_0|_\infty^2|\partial_z\varphi_1|_4^2 y_\varepsilon + C\varepsilon^{\frac{10}{3}}|\varphi_0|_\infty^{\frac83}|\partial_z\varphi_1|_4^{\frac83} y_\varepsilon\\
&+ C\varepsilon^{\frac{16}{7}}|\varphi_0|_\infty^{\frac{16}{7}}|\varphi_1|_4^{\frac{16}{7}} y_\varepsilon + C\varepsilon^4|\varphi_0|_\infty^4|\varphi_1|_4^4 y_\varepsilon + C\varepsilon^{\frac{12}{7}}|\varphi_0|_\infty^{\frac{16}{7}} y_\varepsilon^{\frac{15}{7}} + C\varepsilon^6|\varphi_0|_\infty^8 y_\varepsilon^5\\
&+ C\varepsilon^5|\varphi_1|_8^4|\partial_z\varphi_1|_4^2 + C\varepsilon^{\frac{32}{7}}|\varphi_1|_8^{\frac{32}{7}} y_\varepsilon + C\varepsilon^8|\varphi_1|_8^8 y_\varepsilon\\
&+ C\varepsilon^{\frac{11}{4}}|\partial_z\varphi_1|_4^2 y_\varepsilon^2 + C\varepsilon^{\frac{22}{5}}|\partial_z\varphi_1|_4^{\frac{16}{5}} y_\varepsilon^{\frac{13}{5}} + C\varepsilon^2 y_\varepsilon^{\frac{23}{7}} + \varepsilon^{14} y_\varepsilon^{17}.
\end{aligned}
$$

This concludes the study of the terms issued from the estimate on $\varepsilon^3|\partial_x R_\varepsilon|_2^2$.

• ***Step 4 - Terms issued from the estimate on*** $\boldsymbol{\varepsilon|\partial_z R_\varepsilon|_2^2}$

We recall that, when we estimate the derivatives of R_ε with respect to x, the transport term does not contribute to the energy estimate.

On the other hand, when we deal with the derivatives with respect to z, the transport term has a contribution that we have to bound. Hence, the first two integrals are controlled by using the coercive terms issued from the L^2 estimate on R_ε in the following way:

$$
\begin{aligned}
|L_1| &\le \varepsilon|R_\varepsilon|_2|\partial_x\partial_z R_\varepsilon|_2 \le |R_\varepsilon|_2(\varepsilon|\partial_x\partial_z R_\varepsilon|_2)\\
&\le y_\varepsilon^{\frac12}z_\varepsilon^{\frac12} \le \frac{1}{50}z_\varepsilon + Cy_\varepsilon,
\end{aligned}
\tag{4.45}
$$

and, integrating by parts, it follows that

$$\begin{aligned}|L_2| &\leq \varepsilon^{\frac{1}{2}}|u_0|_\infty(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_2)|\partial_z^2 R_\varepsilon|_2 \leq \varepsilon^{\frac{1}{2}}|u_0|_\infty y_\varepsilon^{\frac{1}{2}} z_\varepsilon^{\frac{1}{2}} \\ &\leq \frac{1}{50}z_\varepsilon + C\varepsilon|u_0|_\infty^2 y_\varepsilon.\end{aligned} \tag{4.46}$$

All the other terms in this estimate can be treated similarly to what precedes. Therefore, we just recall the results we obtain without entering into the details of the computations.

The next fourth terms are obtained by replacing formally $\varepsilon^2\partial_x^2 R_\varepsilon$ by $\partial_z^2 R_\varepsilon$ respectively in (4.38), (4.39), (4.40) and (4.41). The final results are the same, that is to say

$$|L_3| \leq \varepsilon|F''(\varphi_0)|_\infty z_\varepsilon, \tag{4.47}$$

$$|L_4| \leq \frac{1}{50}z_\varepsilon + \varepsilon|F''(\varphi_0)|_\infty z_\varepsilon + C\varepsilon|\partial_z F''(\varphi_0)|_\infty^2 y_\varepsilon + C\varepsilon^2|\partial_z^2 F''(\varphi_0)|_\infty^2 y_\varepsilon, \tag{4.48}$$

$$|L_5| \leq \frac{1}{50}z_\varepsilon + C\varepsilon^{\frac{20}{3}} y_\varepsilon^{\frac{11}{3}} + C\varepsilon^{\frac{5}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon + C\varepsilon^{11}|\partial_x\varphi_1|_4^4 y_\varepsilon + C\varepsilon^7|\partial_z\varphi_1|_4^4 y_\varepsilon, \tag{4.49}$$

and finally

$$|L_6| \leq \frac{1}{50}z_\varepsilon + C\varepsilon^2|\partial_z\varphi_0|_\infty^2 y_\varepsilon. \tag{4.50}$$

As far as the seventh term is concerned, the bound is identical to (4.43):

$$\begin{aligned}|L_7| \leq \frac{1}{50}z_\varepsilon &+ C\varepsilon^5|\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_x\varphi_1|_4^2 + C\varepsilon^{\frac{9}{2}}|\varphi_0|_\infty^2|\partial_x\varphi_1|_4^2 y_\varepsilon + C\varepsilon^6|\varphi_0|_\infty^{\frac{8}{3}}|\partial_x\varphi_1|_4^{\frac{8}{3}} y_\varepsilon \\ &+ C\varepsilon^{\frac{16}{7}}|\varphi_0|_\infty^{\frac{16}{7}}|\varphi_1|_4^{\frac{16}{7}} y_\varepsilon + C\varepsilon^4|\varphi_0|_\infty^4|\varphi_1|_4^4 y_\varepsilon + C\varepsilon^{\frac{12}{7}}|\varphi_0|_\infty^{\frac{16}{7}} y_\varepsilon^{\frac{15}{7}} + C\varepsilon^6|\varphi_0|_\infty^8 y_\varepsilon^5 \\ &+ C\varepsilon^7|\varphi_1|_8^4|\partial_x\varphi_1|_4^2 + C\varepsilon^{\frac{32}{7}}|\varphi_1|_8^{\frac{32}{7}} y_\varepsilon + C\varepsilon^8|\varphi_1|_8^8 y_\varepsilon \\ &+ C\varepsilon^{\frac{19}{4}}|\partial_x\varphi_1|_4^2 y_\varepsilon^2 + C\varepsilon^{\frac{38}{5}}|\partial_x\varphi_1|_4^{\frac{16}{5}} y_\varepsilon^{\frac{13}{5}} + C\varepsilon^2 y_\varepsilon^{\frac{23}{7}} + C\varepsilon^{14} y_\varepsilon^{17}.\end{aligned} \tag{4.51}$$

And finally, the last term gives the same bound as (4.44), namely

$$\begin{aligned}|L_8| \leq \frac{1}{50}z_\varepsilon &+ C\varepsilon^3|\partial_z\varphi_0|_\infty^2|\varphi_1|_4^4 + C\varepsilon^2|\partial_z\varphi_0|_\infty^2 y_\varepsilon + C\varepsilon^{\frac{8}{3}}|\partial_z\varphi_0|_\infty^{\frac{8}{3}} y_\varepsilon \\ &+ C\varepsilon^3|\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_z\varphi_1|_4^2 + C\varepsilon^{\frac{5}{2}}|\varphi_0|_\infty^2|\partial_z\varphi_1|_4^2 y_\varepsilon + C\varepsilon^{\frac{10}{3}}|\varphi_0|_\infty^{\frac{8}{3}}|\partial_z\varphi_1|_4^{\frac{8}{3}} y_\varepsilon \\ &+ C\varepsilon^{\frac{16}{7}}|\varphi_0|_\infty^{\frac{16}{7}}|\varphi_1|_4^{\frac{16}{7}} y_\varepsilon + C\varepsilon^4|\varphi_0|_\infty^4|\varphi_1|_4^4 y_\varepsilon + C\varepsilon^{\frac{12}{7}}|\varphi_0|_\infty^{\frac{16}{7}} y_\varepsilon^{\frac{15}{7}} + C\varepsilon^6|\varphi_0|_\infty^8 y_\varepsilon^5 \\ &+ C\varepsilon^5|\varphi_1|_8^4|\partial_z\varphi_1|_4^2 + C\varepsilon^{\frac{32}{7}}|\varphi_1|_8^{\frac{32}{7}} y_\varepsilon + C\varepsilon^8|\varphi_1|_8^8 y_\varepsilon \\ &+ C\varepsilon^{\frac{11}{4}}|\partial_z\varphi_1|_4^2 y_\varepsilon^2 + C\varepsilon^{\frac{22}{5}}|\partial_z\varphi_1|_4^{\frac{16}{5}} y_\varepsilon^{\frac{13}{5}} + C\varepsilon^2 y_\varepsilon^{\frac{23}{7}} + \varepsilon^{14} y_\varepsilon^{17}.\end{aligned} \tag{4.52}$$

• ***Step 5 - Final estimate***

We gather the previous inequalities all together and we obtain:

$$
\begin{aligned}
&\frac{d}{dt}y_\varepsilon + \frac{1}{2}z_\varepsilon \leq \\
&I_1, ..., I_6 \left\{ \begin{array}{l} \quad C\varepsilon y_\varepsilon z_\varepsilon + C\varepsilon^2|\varphi_0|_\infty^2|\varphi_1|_4^4 + C\varepsilon^4|\varphi_1|_6^6 \\ +C|F''(\varphi_0)|_\infty^2 y_\varepsilon + |\partial_z\varphi_0|_\infty y_\varepsilon + C\varepsilon^{\frac{8}{7}}|\varphi_1|_2^{\frac{16}{7}} y_\varepsilon + C\varepsilon^{\frac{4}{3}}|\varphi_1|_2^{\frac{8}{3}} y_\varepsilon \\ +C\varepsilon|\varphi_0|_\infty^2 y_\varepsilon^2 + C\varepsilon^2(1+|\varphi_0|_\infty^4)y_\varepsilon^3 \end{array} \right. \\
&J_1, ..., J_8 \left\{ \begin{array}{l} +C\varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon + C|\partial_z^2\varphi_0|_\infty^{\frac{4}{3}} y_\varepsilon + C|\partial_z\varphi_0|_\infty^2 y_\varepsilon + C\varepsilon^{\frac{4}{11}}(|\partial_z^2\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{16}{11}} y_\varepsilon \\ +C\varepsilon^{\frac{1}{2}}(|\partial_z^2\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^2 y_\varepsilon + C\varepsilon^3(\varepsilon^4|\partial_x\varphi_1|_4^4 + |\partial_z\varphi_1|_4^4)y_\varepsilon \\ +C|\partial_z^2\varphi_0|_\infty(\varepsilon|\partial_x\varphi_1|_2 + |\partial_z\varphi_1|_2)y_\varepsilon^{\frac{1}{2}} + C|\partial_z\varphi_0|_\infty(\varepsilon^2|\partial_x^2\varphi_1|_2 + |\partial_z^2\varphi_1|_2)y_\varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{4}{3}} y_\varepsilon^{\frac{11}{3}} \\ +C(\varepsilon^{\frac{7}{3}}|\partial_z^2\varphi_1|_2^{\frac{4}{3}}|\partial_x\varphi_1|_4^{\frac{4}{3}} + \varepsilon^{\frac{15}{3}}|\partial_x^2\varphi_1|_2^{\frac{4}{3}}|\partial_x\varphi_1|_4^{\frac{4}{3}} + \varepsilon|\partial_z^2\varphi_1|_2^{\frac{4}{3}}|\partial_z\varphi_1|_4^{\frac{4}{3}} + \varepsilon^{\frac{11}{3}}|\partial_x^2\varphi_1|_2^{\frac{4}{3}}|\partial_z\varphi_1|_4^{\frac{4}{3}})y_\varepsilon^{\frac{1}{3}} \end{array} \right. \\
&\begin{array}{r} K_1, ..., K_6 \\ \text{and } L_3, ..., L_8 \end{array} \left\{ \begin{array}{l} +C\varepsilon|F''(\varphi_0)|_\infty z_\varepsilon + C\varepsilon^3|\partial_z\varphi_0|_\infty^2|\varphi_1|_4^4 \\ +C\varepsilon^3|\varphi_0|_\infty^2|\varphi_1|_4^2(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2) + C\varepsilon^5|\varphi_1|_8^4(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2) \\ +C(\varepsilon|\partial_z F''(\varphi_0)|_\infty^2 + \varepsilon^2|\partial_z^2 F''(\varphi_0)|_\infty^2 + \varepsilon^2|\partial_z\varphi_0|_\infty^2 + \varepsilon^{\frac{8}{3}}|\partial_z\varphi_0|_\infty^{\frac{8}{3}})y_\varepsilon \\ +C\varepsilon^{\frac{5}{2}}|\varphi_0|_\infty^2(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2)y_\varepsilon + C\varepsilon^{\frac{16}{7}}|\varphi_0|_\infty^{\frac{16}{7}}|\varphi_1|_4^{\frac{16}{7}} y_\varepsilon + C\varepsilon^4|\varphi_0|_\infty^4|\varphi_1|_4^4 y_\varepsilon \\ +C\varepsilon^{\frac{10}{3}}|\varphi_0|_\infty^{\frac{8}{3}}(\varepsilon|\partial_x\varphi_1|_4 + |\partial_z\varphi_1|_4)^{\frac{8}{3}} y_\varepsilon + C\varepsilon^{\frac{32}{7}}|\varphi_1|_8^{\frac{32}{7}} y_\varepsilon + C\varepsilon^8|\varphi_1|_8^8 y_\varepsilon \\ +C\varepsilon^{\frac{11}{4}}(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2)y_\varepsilon^2 + C\varepsilon^{\frac{22}{5}}(\varepsilon|\partial_x\varphi_1|_4 + |\partial_z\varphi_1|_4)^{\frac{16}{5}} y_\varepsilon^{\frac{13}{5}} \\ +C\varepsilon^{\frac{12}{7}}|\varphi_0|_\infty^{\frac{16}{7}} y_\varepsilon^{\frac{15}{7}} + C\varepsilon^6|\varphi_0|_\infty^8 y_\varepsilon^5 + C\varepsilon^2 y_\varepsilon^{\frac{23}{7}} + C\varepsilon^{14} y_\varepsilon^{17} \end{array} \right. \\
&L_1 \text{ and } L_2 \left\{ \; +Cy_\varepsilon + C\varepsilon|u_0|_\infty^2 y_\varepsilon. \right.
\end{aligned}
$$

The term in $F''(\varphi_0)$ comes from the Cahn-Hilliard operator and is characteristic of the fact that we do not necessarily choose a reference solution φ_0 which is a constant and metastable state of the potential F. Moreover, the terms which contain some derivatives of φ_0 with respect to z shows that it is the non-uniformity of the solution φ_0 considered here which acts as a source term in the equation on the remainders $R_\varepsilon, W_\varepsilon$ through the capillary force terms.

For convenience, this energy estimate can be written in the following form:

$$
\begin{aligned}
\frac{d}{dt}y_\varepsilon + \frac{1}{2}z_\varepsilon \leq{}& C\varepsilon|F''(\varphi_0)|_\infty z_\varepsilon + C(\varepsilon y_\varepsilon + \varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}})z_\varepsilon + g_1(t) + g_2(t)y_\varepsilon^{\frac{1}{3}} + g_3(t)y_\varepsilon^{\frac{1}{2}} \\
&+ g_4(t)(y_\varepsilon^2 + y_\varepsilon^{\frac{15}{7}} + y_\varepsilon^{\frac{13}{5}} + y_\varepsilon^3 + y_\varepsilon^{\frac{23}{7}} + y_\varepsilon^{\frac{11}{3}} + y_\varepsilon^5 + y_\varepsilon^{17}) + g_5(t)y_\varepsilon,
\end{aligned}
\tag{4.53}
$$

where

$$
\begin{aligned}
g_1(t) ={}& C\varepsilon^2|\varphi_0|_\infty^2|\varphi_1|_4^4 + C\varepsilon^3|\partial_z\varphi_0|_\infty^2|\varphi_1|_4^4 + C\varepsilon^4|\varphi_1|_6^6 \\
&+ C\varepsilon^3|\varphi_0|_\infty^2|\varphi_1|_4^2(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2) + C\varepsilon^5|\varphi_1|_8^4(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2),
\end{aligned}
$$

$$
g_2(t) = C\varepsilon^{\frac{7}{3}}|\partial_z^2\varphi_1|_2^{\frac{4}{3}}|\partial_x\varphi_1|_4^{\frac{4}{3}} + C\varepsilon^{\frac{15}{3}}|\partial_x^2\varphi_1|_2^{\frac{4}{3}}|\partial_x\varphi_1|_4^{\frac{4}{3}} + C\varepsilon|\partial_z^2\varphi_1|_2^{\frac{4}{3}}|\partial_z\varphi_1|_4^{\frac{4}{3}} + C\varepsilon^{\frac{11}{3}}|\partial_x^2\varphi_1|_2^{\frac{4}{3}}|\partial_z\varphi_1|_4^{\frac{4}{3}},
$$

$$
g_3(t) = C|\partial_z^2\varphi_0|_\infty(\varepsilon|\partial_x\varphi_1|_2 + |\partial_z\varphi_1|_2) + C|\partial_z\varphi_0|_\infty(\varepsilon^2|\partial_x^2\varphi_1|_2 + |\partial_z^2\varphi_1|_2),
$$

$$
\begin{aligned}
g_4(t) ={}& C\varepsilon|\varphi_0|_\infty^2 + C\varepsilon^2|\varphi_0|_\infty^4 + C\varepsilon^{\frac{4}{3}} + C\varepsilon^{\frac{12}{7}}|\varphi_0|_\infty^{\frac{16}{7}} + C\varepsilon^6|\varphi_0|_\infty^8 \\
&+ C\varepsilon^{\frac{11}{4}}(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2) + C\varepsilon^{\frac{22}{5}}(\varepsilon|\partial_x\varphi_1|_4 + |\partial_z\varphi_1|_4)^{\frac{16}{5}},
\end{aligned}
$$

$$
\begin{aligned}
g_5(t) &= C|F''(\varphi_0)|_\infty^2 + |\partial_z\varphi_0|_\infty + C\varepsilon^{\frac{8}{7}}|\varphi_1|_2^{\frac{16}{7}} + C\varepsilon^{\frac{4}{3}}|\varphi_1|_2^{\frac{8}{3}} \\
&\quad + C|\partial_z^2\varphi_0|_\infty^{\frac{4}{3}} + C|\partial_z\varphi_0|_\infty^2 + C\varepsilon^{\frac{4}{11}}(|\partial_z^2\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{16}{11}} \\
&\quad + C\varepsilon^{\frac{1}{2}}(|\partial_z^2\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^2 + C\varepsilon^3(\varepsilon^4|\partial_x\varphi_1|_4^4 + |\partial_z\varphi_1|_4^4) \\
&\quad + C(\varepsilon|\partial_z F''(\varphi_0)|_\infty^2 + \varepsilon^2|\partial_z^2 F''(\varphi_0)|_\infty^2 + \varepsilon^{\frac{8}{3}}|\partial_z\varphi_0|_\infty^{\frac{8}{3}}) \\
&\quad + C\varepsilon^{\frac{5}{2}}|\varphi_0|_\infty^2(\varepsilon^2|\partial_x\varphi_1|_4^2 + |\partial_z\varphi_1|_4^2) + C\varepsilon^{\frac{16}{7}}|\varphi_0|_\infty^{\frac{16}{7}}|\varphi_1|_4^{\frac{16}{7}} + C\varepsilon^4|\varphi_0|_\infty^4|\varphi_1|_4^4 \\
&\quad + C\varepsilon^{\frac{10}{3}}|\varphi_0|_\infty^{\frac{8}{3}}(\varepsilon|\partial_x\varphi_1|_4 + |\partial_z\varphi_1|_4)^{\frac{8}{3}} + C\varepsilon^{\frac{32}{7}}|\varphi_1|_8^{\frac{32}{7}} + C\varepsilon^8|\varphi_1|_8^8 \\
&\quad + C + C\varepsilon|u_0|_\infty^2.
\end{aligned}
$$

5 Proofs of theorems 2.1 and 2.2

In this section, we deal with the proofs of theorems 2.1 and 2.2, which concern respectively the general case (φ_0 is any 1D solution of the Cahn-Hilliard equation) and the non-metastable homogeneous case (φ_0 is a constant non-metastable solution).

First of all, it is necessary to establish some bounds on the functions g_i in $L^1_{loc}(\mathbb{R}^+)$, using the estimates on φ_0, φ_1 and u_0 obtained at the beginning of the paper.

• ***Step 1 - Bound for g_1***

Using the anisotropic Sobolev inequality (3.13) and estimates (3.3), (3.16) and (3.19), we have

$$
\begin{aligned}
\varepsilon\int_0^T |\varphi_0|_\infty^2|\varphi_1|_4^4 dt &\le C_0\int_0^T |\varphi_1|_2^3(\varepsilon^2|\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{2}}(|\varphi_1|_2 + |\partial_z\varphi_1|_2)^{\frac{1}{2}} dt \\
&\le C_0|\varphi_1^0|_2^4 e^{C_0T},
\end{aligned}
$$

which implies, in view of (3.2),

$$
\begin{aligned}
\varepsilon\int_0^T |\partial_z\varphi_0|_\infty^2|\varphi_1|_4^4 dt &\le \int_0^T |\partial_z\varphi_0|_2^{\frac{3}{2}}|\partial_z^3\varphi_0|_2^{\frac{1}{2}}|\varphi_1|_2^3(\varepsilon^2|\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{2}}(|\varphi_1|_2 + |\partial_z\varphi_1|_2)^{\frac{1}{2}} dt \\
&\le C_0(1+T)|\varphi_1^0|_2^4 e^{C_0T}.
\end{aligned}
$$

In the same way, with the Sobolev inequality (3.13), we have

$$
\begin{aligned}
\varepsilon^2\int_0^T |\varphi_1|_6^6 dt &\le C\int_0^T |\varphi_1|_2^4(\varepsilon^2|\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)(|\varphi_1|_2 + |\partial_z^2\varphi_1|_2) dt \\
&\le |\varphi_1^0|_2^4 e^{2C_0T}\int_0^T (\varepsilon^2|\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)(|\varphi_1|_2 + |\partial_z^2\varphi_1|_2) dt \\
&\le |\varphi_1^0|_2^6 e^{3C_0T}.
\end{aligned}
$$

Using the two inequalities (3.11) and (3.13), one has

$$
\begin{aligned}
\varepsilon^3\int_0^T |\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_x\varphi_1|_4^2 dt &\le C_0\int_0^T |\varphi_1|_2^{\frac{3}{2}}(\varepsilon^2|\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{4}}(|\varphi_1|_2 + |\partial_z^2\varphi_1|_2)^{\frac{1}{4}} \\
&\qquad \times |\varphi_1|_2^{\frac{1}{2}}(\varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{2}}(\varepsilon^2|\varphi_1|_2 + \varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{2}} \\
&\qquad \times \left(|\varphi_1|_2^{\frac{1}{2}}(\varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{2}} + |\partial_z^2\varphi_1|_2^{\frac{1}{2}}(\varepsilon^2|\partial_x^2\varphi_1|_2)^{\frac{1}{2}}\right)^{\frac{1}{2}} dt \\
&\le C_0|\varphi_1^0|_2^4 e^{C_0T},
\end{aligned}
$$

and furthermore

$$\varepsilon \int_0^T |\varphi_0|_\infty^2 |\varphi_1|_4^2 |\partial_z \varphi_1|_4^2 dt \leq C_0 |\varphi_1^0|_2^4 e^{C_0 T}.$$

Finally, always using anisotropic Sobolev inequalities, it follows that

$$\begin{aligned}\varepsilon^5 \int_0^T |\varphi_1|_8^4 |\partial_x \varphi_1|_4^2 dt &\leq \int_0^T |\varphi_1|_2^{\frac{5}{2}} (\varepsilon^2 |\varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{3}{4}} (|\varphi_1|_2 + |\partial_z^2 \varphi_1|_2)^{\frac{3}{4}} \\ &\quad \times (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1|_2)^{\frac{3}{2}} (\varepsilon^{\frac{7}{2}} |\partial_x \varphi_1|_2 + \varepsilon^{\frac{7}{2}} |\partial_x^3 \varphi_1|_2)^{\frac{1}{4}} (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1|_2 + \varepsilon^{\frac{3}{2}} |\partial_x \partial_z^2 \varphi_1|_2)^{\frac{1}{4}} dt \\ &\leq |\varphi_1^0|_2^4 (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2)^2 e^{3C_0 T},\end{aligned}$$

and thus

$$\varepsilon^3 \int_0^T |\varphi_1|_8^4 |\partial_z \varphi_1|_4^2 dt \leq |\varphi_1^0|_2^4 (\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + |\varphi_1^0|_2)^2 e^{3C_0 T}.$$

- ***Step 2 - Bound for g_2***

All the terms in g_2 are similar and can be bounded, for instance, in the following way:

$$\begin{aligned}\varepsilon^2 \int_0^T |\partial_z^2 \varphi_1|_2^{\frac{4}{3}} |\partial_x \varphi_1|_4^{\frac{4}{3}} dt &\leq \int_0^T |\partial_z^2 \varphi_1|_2^{\frac{4}{3}} (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1|_2)^{\frac{2}{3}} (\varepsilon^2 |\partial_x \varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{3}} (\varepsilon |\partial_x \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2)^{\frac{1}{3}} dt \\ &\leq |\varphi_1^0|_2^2 (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2)^{\frac{2}{3}} e^{\frac{4}{3} C_0 T},\end{aligned}$$

which implies

$$\int_0^T g_2 dt \leq C_0 \varepsilon^{\frac{1}{3}} |\varphi_1^0|_2^2 (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2 + \varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + |\varphi_1^0|_2)^{\frac{2}{3}} e^{C_0 T}.$$

- ***Step 3 - Bound for g_3***

Using estimates (3.1) and (3.2) on φ_0 and the embedding of H^1 in L^∞ in 1D, it follows that

$$\begin{aligned}\int_0^T g_3(t) dt &\leq C \int_0^T |\partial_z^3 \varphi_0|_2 |\varphi_1|_2^{\frac{1}{2}} ((\varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} + |\partial_z^2 \varphi_1|_2^{\frac{1}{2}}) dt + C \int_0^T |\partial_z^3 \varphi_0|_2 (\varepsilon^2 |\partial_x^2 \varphi_1|_2 + |\partial_z^2 \varphi_1|_2) dt \\ &\leq C_0 |\varphi_1^0|_2 \sqrt{1+T} e^{C_0 T} \\ &\leq C_0 |\varphi_1^0|_2 e^{C_0 T}.\end{aligned}$$

- ***Step 4 - Bound for g_4***

The five first terms are easily controlled in $L^1(0,T)$ by $C_0 \varepsilon T$.
Inequality (3.11) and lemma 3.2, leads to

$$\begin{aligned}\varepsilon^{\frac{5}{2}} \int_0^T |\partial_x \varphi_1|_4^2 dt &\leq |\varphi_1|_2^{\frac{1}{2}} (\varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^2 |\partial_x \varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} (\varepsilon |\partial_x \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2)^{\frac{1}{2}} \\ &\leq C |\varphi_1^0|_2^2 e^{C_0 T},\end{aligned}$$

and

$$\varepsilon^{\frac{1}{2}} \int_0^T |\partial_z \varphi_1|_4^2 dt \leq C |\varphi_1^0|_2^2 e^{C_0 T}.$$

In the same way, one has

$$\begin{aligned}\varepsilon^{\frac{24}{5}} \int_0^T |\partial_x \varphi_1|_4^{\frac{16}{5}} dt &\leq C \int_0^T (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1|_2)^{\frac{8}{5}} (\varepsilon^2 |\partial_x \varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{4}{5}} (\varepsilon |\partial_x \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2)^{\frac{4}{5}} dt \\ &\leq C (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2)^{\frac{8}{5}} |\varphi_1^0|_2^{\frac{8}{5}} e^{C_0 T},\end{aligned}$$

and

$$\varepsilon^{\frac{8}{5}} \int_0^T |\partial_z \varphi_1|_4^{\frac{16}{5}} dt \leq C(\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + |\varphi_1^0|_2)^{\frac{8}{5}} |\varphi_1^0|_2^{\frac{8}{5}} e^{C_0 T}.$$

• ***Step 5 - Bound for g_5***

Most of the terms can be bounded in the very same way as the previous one. The only terms which need an additional treatment are the following.

With (3.11) it follows

$$\begin{aligned}\varepsilon^2 \int_0^T |\partial_z \varphi_1|_4^4 dt &\leq C \int_0^T (\varepsilon |\partial_z \varphi_1|_2^2)(\varepsilon |\partial_z \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2)(|\varphi_1|_2 + |\partial_z^2 \varphi_1|_2) dt \\ &\leq C(\varepsilon |\partial_z \varphi_1^0|_2^2 + C_0 |\varphi_1^0|_2^2) |\varphi_1^0|_2^2 e^{C_0 T},\end{aligned}$$

and

$$\varepsilon^6 \int_0^T |\partial_x \varphi_1|_4^4 dt \leq C(\varepsilon^3 |\partial_x \varphi_1^0|_2^2) |\varphi_1^0|_2^2 e^{C_0 T}.$$

Finally, thanks to the Sobolev inequality (3.13) with $p = 8$, one has

$$\begin{aligned}\varepsilon^{\frac{9}{2}} \int_0^T |\varphi_1|_8^8 dt &\leq C \varepsilon^{\frac{9}{2}} \int_0^T |\varphi_1|_2^5 (|\varphi_1|_2 + |\partial_x^2 \varphi_1|_2)^{\frac{3}{2}} (|\varphi_1|_2 + |\partial_z^2 \varphi_1|_2)^{\frac{3}{2}} dt \\ &\leq C \int_0^T |\varphi_1|_2^5 \Big(|\varphi_1|_2 + (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^{\frac{7}{2}} |\partial_x^3 \varphi_1|_2)^{\frac{1}{2}} \Big)^{\frac{3}{2}} \Big(|\varphi_1|_2 + (\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^{\frac{1}{2}} |\partial_z^3 \varphi_1|_2)^{\frac{1}{2}} \Big)^{\frac{3}{2}} dt \\ &\leq C_0 |\varphi_1^0|_2^5 (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2)^{\frac{3}{2}} (\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + |\varphi_1^0|_2)^{\frac{3}{2}} e^{C_0 T}.\end{aligned}$$

• ***Step 6 - End of the proofs***

We can now prove the first two theorems stated in section 2.

Proof (of theorem 2.1):

Assumption (2.7) on $\tilde{\varphi}_1^0$ can be written in the new variables (defined on Ω_0) in the following form:

$$|\varphi_1^0|_2 + \varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + \varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2 \leq K_0.$$

Then, the previous bounds on $g_1, ..., g_5$ show that

$$\int_0^T g_1 dt \leq \varepsilon K_0' e^{C_0 T}, \;\; \int_0^T g_2 dt \leq \varepsilon^{\frac{1}{3}} K_0' e^{C_0 T}, \;\; \int_0^T g_3 dt \leq K_0' e^{C_0 T},$$

$$\int_0^T g_4 dt \leq \varepsilon K_0' e^{C_0 T}, \;\; \int_0^T g_5 dt \leq C_0 T + K_0' e^{C_0 T},$$

where K_0' is a constant depending only on φ_0 and K_0.

As previously noted, we can see at this step of the proof that two kinds of terms prevent us from obtaining significant results on large times (on $\mathbb{R}^+$ for instance).

- On the one hand, the term g_3, issued from the inhomogeneity in z of φ_0, appears as a source term in the equation on W_ε coming from the capillary forces. In other words, the capillary forces created by the non-homogeneity of φ_0 are the main reason for the increase of W_ε with the time. Theorem 2.2 is a confirmation of this fact.
- On the other hand, the fact that $F''(\varphi_0)$ is not necessarily positive is also a source of increase for the perturbations due to the instability of such a solution for the 1D Cahn-Hilliard equation. Theorem 2.3 shows that, if we indeed assume that φ_0 is constant and metastable, then the persistency of the perturbation is justified for all time.

Let us introduce

$$M_\varepsilon(T) = 2(y_\varepsilon(0) + C_0 e^{C_0 T})e^{C_0 T + C_0 e^{C_0 T}}.$$

As $R_\varepsilon(0) = 0$ and $W_\varepsilon(0) = W_\varepsilon^0$, assumption (2.8) shows that

$$\forall \varepsilon > 0, \ \ y_\varepsilon(0) \le K_0^2.$$

Hence, there exists $M_0(T)$ depending only on φ_0, u_0, K_0 and T such that

$$\forall \varepsilon > 0, \forall T > 0, \ \ M_\varepsilon(T) \le M_0(T).$$

For a fixed time $T > 0$, as $M_\varepsilon(T)$ is uniformly bounded with respect to ε, it is clear that there exists $\varepsilon_0(T)$ such that, for any $\varepsilon < \varepsilon_0$, we have

$$C(M_\varepsilon(T) + (M_\varepsilon(T))^{16}) \le \frac{1}{\sqrt{\varepsilon}}, \ \text{ and } \ (\varepsilon M_\varepsilon(T) + \varepsilon^{\frac{1}{4}} M_\varepsilon(T)^{\frac{1}{2}}) \le \frac{1}{4}.$$

Let us suppose, furthermore, that ε_0 is small enough to insure

$$C\varepsilon_0 |F''(\varphi_0)|_\infty \le \frac{1}{4}.$$

Let $[0, T^*]$ be the maximal time interval in $[0, T]$ such that, for any $t \in [0, T^*]$,

$$y_\varepsilon(t) \le M_\varepsilon(T).$$

On this interval, using Young's inequality and the previous properties of $M_\varepsilon(T)$ which let us absorb the terms in z_ε which are present in the right-hand side of inequality (4.53), there remains

$$\frac{d}{dt} y_\varepsilon \le g_1(t) + g_2(t) + g_3(t) + \left(\frac{1}{\sqrt{\varepsilon}} g_4(t) + g_2(t) + g_3(t) + g_5(t)\right) y_\varepsilon(t).$$

Then, for any $t \in [0, T^*] \subset [0, T]$,

$$y_\varepsilon(t) \le y_\varepsilon(0) + \int_0^T (g_1 + g_2 + g_3) ds + \int_0^t \left(g_2 + g_3 + \frac{1}{\sqrt{\varepsilon}} g_4 + g_5\right) y_\varepsilon ds,$$

so that, by Gronwall's lemma, using the bounds in $L^1(0, T)$ on the functions g_i, it follows that

$$y_\varepsilon(t) \le (y_\varepsilon(0) + C_0 e^{C_0 T}) e^{C_0 e^{C_0 T}} \le \frac{1}{2} M_\varepsilon(T),$$

which implies, by a continuity argument, that $T^* = T$. Therefore, we finally have

$$\sup_{t \in [0,T]} y_\varepsilon(t) \le M_0(T),$$

for any $\varepsilon < \varepsilon_0(T)$.

This gives the desired result using (4.14), that is to say the definition of the energy y_ε in the initial variables. ■

Proof (of theorem 2.2):

We suppose now that the 1D solution of the Cahn-Hilliard equation under study is constant $\varphi_0 = \omega$ independent of t, x and z, that is to say that we consider the physical case of an homogeneous initial state of the system. We suppose furthermore that this homogeneous mixture is not a metastable state of the Cahn-Hilliard potential, that is to say that $F''(\omega) < 0$.

In that case, some of the terms in the previous estimates vanish. Indeed, we have seen above that, when φ_0 is not constant in the previous theorem, it introduces some source terms in the equations on the remainders of the asymptotic expansion.

More precisely, the term $g_3(t)$ in estimate (4.53) vanishes, as well as some terms in $g_5(t)$. Then, with the assumptions on φ_1^0 given by (2.9), it follows that

$$\int_0^T g_1 dt \leq \varepsilon C_0 e^{C_0 T}, \ \int_0^T g_2 dt \leq \varepsilon^{\frac{1}{3}} C_0 e^{C_0 T},$$

$$g_3 = 0,$$

$$\int_0^T g_4 dt \leq \varepsilon C_0 e^{C_0 T}, \ \int_0^T g_5 dt \leq C_0 T + C_0 e^{C_0 T}.$$

Of course, the term coming from the non-convexity of F in the neighborhood of ω is still present in $g_5(t)$ and always prevent us from obtaining a uniform result on large times. Nevertheless, as g_3 disappears in the estimates, we can improve the result of theorem 2.1.

Consider this time

$$M_\varepsilon(T) = 2(y_\varepsilon(0) + \varepsilon^{\frac{1}{2}} C_0 e^{C_0 T}) e^{C_2 T + C_0 e^{C_0 T}}.$$

As $R_\varepsilon(0) = 0$ and $W_\varepsilon(0)$ satisfies (2.10), we have for any $\varepsilon > 0$

$$y_\varepsilon(0) \leq \varepsilon^{\frac{1}{2}} K_0^2,$$

which implies that there exists $M_0(T)$ depending only on K_0, φ_0, u_0 and T such that

$$\forall \varepsilon > 0, \ \forall T > 0, \ \ M_\varepsilon(T) \leq \varepsilon^{\frac{1}{2}} M_0(T).$$

Hence, if $T > 0$ is given, there exists $\varepsilon_0(T)$ such that, for any $\varepsilon < \varepsilon_0$, we have

$$C(M_\varepsilon(T) + (M_\varepsilon(T))^{16}) \leq 1, \ \text{ and } \ (\varepsilon M_\varepsilon(T) + \varepsilon^{\frac{1}{4}} M_\varepsilon(T)^{\frac{1}{2}}) \leq \frac{1}{4},$$

and we can impose that ε_0 be small enough so that

$$C\varepsilon_0 |F''(\varphi_0)|_\infty \leq \frac{1}{4}.$$

Let $[0, T^*]$ be the maximal interval in $[0, T]$ on which we have

$$y_\varepsilon(t) \leq M_\varepsilon(T).$$

Using Young's inequality, it follows that

$$\varepsilon^{\frac{1}{3}} y_\varepsilon^{\frac{1}{3}} \leq C(\varepsilon^{\frac{1}{2}} + y_\varepsilon),$$

and absorbing the terms in z_ε in estimate (4.53) finally gives

$$\frac{d}{dt} y_\varepsilon \leq g_1(t) + \varepsilon^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2(t) \right) + \left(g_4(t) + \frac{1}{\varepsilon^{\frac{1}{3}}} g_2(t) + g_5(t) \right) y_\varepsilon(t).$$

Therefore, after integration in time, one has, for any $t \in [0, T^*] \subset [0, T]$,

$$y_\varepsilon(t) \leq y_\varepsilon(0) + \int_0^T \left(g_1 + \varepsilon^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2 \right) \right) ds + \int_0^t \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2 + g_4 + g_5 \right) y_\varepsilon ds,$$

so that by Gronwall's lemma it follows that

$$y_\varepsilon(t) \leq (y_\varepsilon(0) + \varepsilon^{\frac{1}{2}} C_0 e^{C_0 T}) e^{C_0 e^{C_0 T}} \leq \frac{1}{2} M_\varepsilon(T).$$

Finally, we conclude that $T^* = T$ and that

$$\sup_{t\in[0,T]} y_\varepsilon(t) \leq \varepsilon^{\frac{1}{2}} M_0(T),$$

for any $\varepsilon < \varepsilon_0(T)$.

This concludes the proof of the theorem. ■

6 Proof of theorem 2.3

This section is concerned with the proof of theorem 2.3.

We suppose once again that $\varphi_0 = \omega$ is a constant solution of the 1D Cahn-Hilliard equation but we make the additional assumption that $F''(\omega) > 0$. As it is well known [1, 6, 7], those particular homogeneous states are stable for the Cahn-Hilliard equation. These states are local minimizers of the Cahn-Hilliard energy under the constraint of constant mass.

Let us recall that, in this particular case, the estimates on φ_1 are different from the ones in the general case (lemma 3.2), they are in particular uniform in time. Moreover, in the main energy estimate that we consider in this paper, the terms I_1, I_2, K_1, K_2, L_3 and L_4 become coercive, so that we do not have to bound these terms any more and, furthermore, the functional z_ε is enforced in

$$\begin{aligned}
z_\varepsilon(t) = {} & \alpha^2\varepsilon^4|\partial_x^2 R_\varepsilon|_2^2 + 2\alpha^2\varepsilon^2|\partial_x\partial_z R_\varepsilon|_2^2 + \alpha^2|\partial_z^2 R_\varepsilon|_2^2 \\
& + \alpha^2\varepsilon^7|\partial_x^3 R_\varepsilon|_2^2 + \alpha^2\varepsilon^5|\partial_x^2\partial_z R_\varepsilon|_2^2 + \alpha^2\varepsilon^3|\partial_x\partial_z^2 R_\varepsilon|_2^2 + \alpha^2\varepsilon|\partial_z^3 R_\varepsilon|_2^2 \\
& + \varepsilon^2 \left|\partial_x\left(\frac{u_\varepsilon}{\varepsilon}\right)\right|_2^2 + \left|\partial_z\left(\frac{u_\varepsilon}{\varepsilon}\right)\right|_2^2 + \varepsilon^2|\partial_x v_\varepsilon|_2^2 + |\partial_z v_\varepsilon|_2^2 \\
& + \varepsilon^2 F''(\omega)|\partial_x R_\varepsilon|_2^2 + F''(\omega)|\partial_z R_\varepsilon|_2^2.
\end{aligned}$$

Indeed, only the terms I_1 and I_2 bring new contributions to the coercive terms, namely $\varepsilon^2|\partial_x R_\varepsilon|_2^2$ and $|\partial_z R_\varepsilon|_2^2$.

We have now to take into account these new fundamental elements in the estimates of the twenty-eight terms I_1, ..., L_8. In particular, thanks to Poincaré inequality (4.16), we have

$$\varepsilon^2|R_\varepsilon|_2^2 \leq C(\varepsilon^2|\partial_x R_\varepsilon|_2^2 + \varepsilon^2|\partial_z R_\varepsilon|_2^2) \leq C z_\varepsilon, \tag{6.1}$$

which allows us to improve the bounds on the norms in L^4, L^6 and L^8 of R_ε in function of y_ε and z_ε, using the anisotropic Sobolev inequalities (3.11) and (3.13) as follows:

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}|R_\varepsilon|_4 & \leq C|R_\varepsilon|_2^{\frac{1}{2}}(\varepsilon|R_\varepsilon|_2 + \varepsilon|\partial_x R_\varepsilon|_2)^{\frac{1}{4}}(\varepsilon|R_\varepsilon|_2 + \varepsilon|\partial_z R_\varepsilon|_2)^{\frac{1}{4}} \\
& \leq C y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{1}{4}},
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}|R_\varepsilon|_6 & \leq C|R_\varepsilon|_2^{\frac{2}{3}}(\varepsilon^2|R_\varepsilon|_2 + \varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac{1}{6}}(\varepsilon|R_\varepsilon|_2 + \varepsilon|\partial_z^2 R_\varepsilon|_2)^{\frac{1}{6}} \\
& \leq C y_\varepsilon^{\frac{1}{3}} z_\varepsilon^{\frac{1}{6}},
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
\varepsilon^{\frac{5}{8}}|R_\varepsilon|_8 \leq {} & C|R_\varepsilon|_2^{\frac{9}{16}}(\varepsilon|R_\varepsilon|_2)^{\frac{1}{16}}\left(\varepsilon^2(\varepsilon|R_\varepsilon|_2)^{\frac{1}{2}}|R_\varepsilon|_2^{\frac{1}{2}} + (\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_2)^{\frac{1}{2}}(\varepsilon^{\frac{7}{2}}|\partial_x^3 R_\varepsilon|_2)^{\frac{1}{2}}\right)^{\frac{3}{16}} \\
& \times\left((\varepsilon|R_\varepsilon|_2)^{\frac{1}{2}}|R_\varepsilon|_2^{\frac{1}{2}} + (\varepsilon^{\frac{1}{2}}|\partial_z R_\varepsilon|_2)^{\frac{1}{2}}(\varepsilon^{\frac{1}{2}}|\partial_z^3 R_\varepsilon|_2)^{\frac{1}{2}}\right)^{\frac{3}{16}} \\
\leq {} & C y_\varepsilon^{\frac{9}{32}} z_\varepsilon^{\frac{1}{32}}(y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{1}{4}})^{\frac{3}{16}}(y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{1}{4}})^{\frac{3}{16}} \\
\leq {} & C y_\varepsilon^{\frac{3}{8}} z_\varepsilon^{\frac{1}{8}}.
\end{aligned} \tag{6.4}$$

We can also improve the bound on the L^4-norm of the derivatives of R_ε in the following way:

$$\begin{aligned}\varepsilon^{\frac{5}{4}}|\partial_x R_\varepsilon|_4 &\le C(\varepsilon|\partial_x R_\varepsilon|_2)^{\frac{1}{2}}(\varepsilon^2|\partial_x R_\varepsilon|_2+\varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac{1}{4}}(\varepsilon|\partial_x R_\varepsilon|_2+\varepsilon|\partial_x\partial_z R_\varepsilon|_2)^{\frac{1}{4}}\\ &\le C z_\varepsilon^{\frac{1}{2}},\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4 &\le C(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_2)^{\frac{1}{2}}(\varepsilon^2|\partial_x R_\varepsilon|_2+\varepsilon^2|\partial_x^2 R_\varepsilon|_2)^{\frac{1}{4}}(\varepsilon|\partial_x R_\varepsilon|_2+\varepsilon|\partial_x\partial_z R_\varepsilon|_2)^{\frac{1}{4}}\\ &\le C y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{1}{4}}.\end{aligned} \tag{6.6}$$

In the same way, one has

$$\varepsilon^{\frac{1}{4}}|\partial_z R_\varepsilon|_4 \le C z_\varepsilon^{\frac{1}{2}}, \tag{6.7}$$

and

$$\varepsilon^{\frac{1}{2}}|\partial_z R_\varepsilon|_4 \le C y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{1}{4}}. \tag{6.8}$$

With the help of these new inequalities, we can improve the bounds on the various terms of the energy estimate.

First of all, the terms I_1 and I_2, as we have already seen, become coercive and so we do not have to estimate them any more. The term I_3, after integration by parts, can be controlled with (6.2) and (6.3) by

$$\begin{aligned}|I_3| &\le \varepsilon^3|\varphi_0|_\infty|\varphi_1|_4|\partial_x\varphi_1|_4|\partial_x R_\varepsilon|_2+\varepsilon^3|\varphi_0|_\infty|R_\varepsilon|_4^2|\partial_x^2 R_\varepsilon|_2+\varepsilon^4|\varphi_1|_6^3|\partial_x^2 R_\varepsilon|_2+\varepsilon^4|R_\varepsilon|_6^3|\partial_x^2 R_\varepsilon|_2\\ &\le \varepsilon^2|\varphi_0|_\infty|\varphi_1|_4|\partial_x\varphi_1|_4(\varepsilon|\partial_x R_\varepsilon|_2)+|\varphi_0|_\infty(\varepsilon^{\frac{1}{2}}|R_\varepsilon|_4)^2(\varepsilon^2|\partial_x^2 R_\varepsilon|_2)\\ &\qquad +\varepsilon^2|\varphi_1|_6^3(\varepsilon^2|\partial_x^2 R_\varepsilon|_2)+\varepsilon^{\frac{1}{2}}(\varepsilon^{\frac{1}{2}}|R_\varepsilon|_6)^3(\varepsilon^2|\partial_x^2 R_\varepsilon|_2)\\ &\le \frac{1}{50}z_\varepsilon+\varepsilon^4|\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_x\varphi_1|_4^2+|\varphi_0|_\infty y_\varepsilon^{\frac{1}{2}} z_\varepsilon+\varepsilon^4|\varphi_1|_6^6+\varepsilon^{\frac{1}{2}} y_\varepsilon z_\varepsilon.\end{aligned} \tag{6.9}$$

And identically for I_4, it follows that

$$|I_4| \le \frac{1}{50}z_\varepsilon+\varepsilon^2|\varphi_0|_\infty^2|\varphi_1|_4^2|\partial_z\varphi_1|_4^2+|\varphi_0|_\infty y_\varepsilon^{\frac{1}{2}} z_\varepsilon+\varepsilon^4|\varphi_1|_6^6+\varepsilon^{\frac{1}{2}} y_\varepsilon z_\varepsilon. \tag{6.10}$$

The term I_5 vanishes because $\partial_z\varphi_0=0$ and, thanks to (4.26)-(4.27), (6.5) and (6.7), we show that

$$|I_6| \le C\varepsilon^{\frac{1}{2}}|\varphi_1|_2 z_\varepsilon. \tag{6.11}$$

Furthermore, the terms J_1, J_2, J_5 and J_6 vanish because of the uniformity of φ_0. The bound on J_3 becomes in those conditions, with (6.6),

$$\begin{aligned}|J_3| &\le \varepsilon^{\frac{7}{4}}|\partial_z^2\varphi_1|_2|\partial_x\varphi_1|_4\left(\varepsilon^{\frac{1}{4}}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)+\varepsilon^{\frac{1}{4}}|\partial_z^2\varphi_1|_2(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4)\left(\varepsilon^{\frac{1}{4}}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)\\ &\qquad +\varepsilon^{\frac{7}{4}}|\partial_z^2 R_\varepsilon|_2|\partial_x\varphi_1|_4\left(\varepsilon^{\frac{1}{4}}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)+\varepsilon^{\frac{1}{4}}|\partial_z^2 R_\varepsilon|_2(\varepsilon^{\frac{3}{2}}|\partial_x R_\varepsilon|_4)\left(\varepsilon^{\frac{1}{4}}\left|\frac{u_\varepsilon}{\varepsilon}\right|_4\right)\\ &\le C\varepsilon^{\frac{7}{4}}|\partial_z^2\varphi_1|_2|\partial_x\varphi_1|_4 y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{1}{4}}+C\varepsilon^{\frac{1}{4}}|\partial_z^2\varphi_1|_2 y_\varepsilon^{\frac{1}{2}} z_\varepsilon^{\frac{1}{2}}+C\varepsilon^{\frac{7}{4}}|\partial_x\varphi_1|_4 y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{3}{4}}+C\varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon^{\frac{1}{2}}\\ &\le \frac{1}{50}z_\varepsilon+C\varepsilon^{\frac{7}{3}}|\partial_z^2\varphi_1|_2^{\frac{4}{3}}|\partial_x\varphi_1|_4^{\frac{4}{3}} y_\varepsilon^{\frac{1}{3}}+\varepsilon^{\frac{1}{2}}|\partial_z^2\varphi_1|_2^2 y_\varepsilon+\varepsilon^7|\partial_x\varphi_1|_4^4 y_\varepsilon+\varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon.\end{aligned} \tag{6.12}$$

In a similar way J_4, J_7 and J_8 are bounded as follows:

$$|J_4| \le \frac{1}{50}z_\varepsilon+C\varepsilon^{\frac{15}{3}}|\partial_x^2\varphi_1|_2^{\frac{4}{3}}|\partial_x\varphi_1|_4^{\frac{4}{3}} y_\varepsilon^{\frac{1}{3}}+\varepsilon^{\frac{9}{2}}|\partial_x^2\varphi_1|_2^2 y_\varepsilon+\varepsilon^7|\partial_x\varphi_1|_4^4 y_\varepsilon+\varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon, \tag{6.13}$$

$$|J_7| \le \frac{1}{50}z_\varepsilon+C\varepsilon|\partial_z^2\varphi_1|_2^{\frac{4}{3}}|\partial_z\varphi_1|_4^{\frac{4}{3}} y_\varepsilon^{\frac{1}{3}}+\varepsilon^{\frac{1}{2}}|\partial_z^2\varphi_1|_2^2 y_\varepsilon+\varepsilon^3|\partial_z\varphi_1|_4^4 y_\varepsilon+\varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon, \tag{6.14}$$

$$|J_8| \le \frac{1}{50} z_\varepsilon + C\varepsilon^{\frac{11}{3}} |\partial_x^2 \varphi_1|_2^{\frac{4}{3}} |\partial_z \varphi_1|_4^{\frac{4}{3}} y_\varepsilon^{\frac{1}{3}} + \varepsilon^{\frac{9}{2}} |\partial_x^2 \varphi_1|_2^2 y_\varepsilon + \varepsilon^3 |\partial_z \varphi_1|_4^4 y_\varepsilon + \varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon. \tag{6.15}$$

The terms K_1, K_2, L_3 and L_4 are now coercive and we do not need to estimate them. The term K_3 is bounded as for J_4 and J_8 and the terms K_4 and L_6 vanish because $\partial_z \varphi_0 = 0$.

The estimate on K_5 can be simplified and becomes

$$\begin{aligned}
|K_5| \le\ & C\varepsilon^{\frac{5}{2}} |\varphi_0|_\infty |\varphi_1|_4 |\partial_x \varphi_1|_4 (\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) + C\varepsilon^2 |\varphi_0|_\infty (\varepsilon^{\frac{1}{2}} |R_\varepsilon|_4) |\partial_x \varphi_1|_4 (\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) \\
& + C\varepsilon |\varphi_0|_\infty |\varphi_1|_4 (\varepsilon^{\frac{3}{2}} |\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) + C\varepsilon^{\frac{1}{2}} |\varphi_0|_\infty (\varepsilon^{\frac{1}{2}} |R_\varepsilon|_4)(\varepsilon^{\frac{3}{2}} |\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) \\
& + C\varepsilon^{\frac{7}{2}} |\varphi_1|_8^2 |\partial_x \varphi_1|_4 (\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) + C\varepsilon^2 |\varphi_1|_8^2 (\varepsilon^{\frac{3}{2}} |\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) \\
& + C\varepsilon^{\frac{9}{4}} (\varepsilon^{\frac{5}{8}} |R_\varepsilon|_8)^2 |\partial_x \varphi_1|_4 (\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) + C\varepsilon^{\frac{3}{4}} (\varepsilon^{\frac{5}{8}} |R_\varepsilon|_8)^2 (\varepsilon^{\frac{3}{2}} |\partial_x R_\varepsilon|_4)(\varepsilon^{\frac{7}{2}} |\partial_x^3 R_\varepsilon|_2) \\
\le\ & C\varepsilon^{\frac{5}{2}} |\varphi_0|_\infty |\varphi_1|_4 |\partial_x \varphi_1|_4 z_\varepsilon^{\frac{1}{2}} + C\varepsilon^2 |\varphi_0|_\infty |\partial_x \varphi_1|_4 y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{3}{4}} \\
& + C\varepsilon |\varphi_0|_\infty |\varphi_1|_4 y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{3}{4}} + C|\varphi_0|_\infty \varepsilon^{\frac{1}{2}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon \\
& + C\varepsilon^{\frac{7}{2}} |\varphi_1|_8^2 |\partial_x \varphi_1|_4 z_\varepsilon^{\frac{1}{2}} + C\varepsilon^2 |\varphi_1|_8^2 y_\varepsilon^{\frac{1}{4}} z_\varepsilon^{\frac{3}{4}} \\
& + C\varepsilon^{\frac{9}{4}} |\partial_x \varphi_1|_4 y_\varepsilon^{\frac{3}{4}} z_\varepsilon^{\frac{3}{4}} + C\varepsilon^{\frac{3}{4}} y_\varepsilon z_\varepsilon \\
\le\ & \frac{1}{50} z_\varepsilon + C\varepsilon^5 |\varphi_0|_\infty^2 |\varphi_1|_4^2 |\partial_x \varphi_1|_4^2 + C\varepsilon^8 |\varphi_0|_\infty^4 |\partial_x \varphi_1|_4^4 y_\varepsilon + C\varepsilon^4 |\varphi_0|_\infty^4 |\varphi_1|_4^4 y_\varepsilon + C|\varphi_0|_\infty \varepsilon^{\frac{1}{2}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon \\
& + C\varepsilon^7 |\varphi_1|_8^4 |\partial_x \varphi_1|_4^2 + C\varepsilon^8 |\varphi_1|_8^8 y_\varepsilon + C\varepsilon^9 |\partial_x \varphi_1|_4^4 y_\varepsilon^3 + C\varepsilon^{\frac{3}{4}} y_\varepsilon z_\varepsilon.
\end{aligned} \tag{6.16}$$

We proceed in the same way for K_6, to get

$$\begin{aligned}
|K_6| \le\ & \frac{1}{50} z_\varepsilon + C\varepsilon^3 |\varphi_0|_\infty^2 |\varphi_1|_4^2 |\partial_z \varphi_1|_4^2 + C\varepsilon^4 |\varphi_0|_\infty^4 |\partial_z \varphi_1|_4^4 y_\varepsilon + C\varepsilon^4 |\varphi_0|_\infty^4 |\varphi_1|_4^4 y_\varepsilon + C|\varphi_0|_\infty \varepsilon^{\frac{1}{2}} y_\varepsilon^{\frac{1}{2}} z_\varepsilon \\
& + C\varepsilon^5 |\varphi_1|_8^4 |\partial_z \varphi_1|_4^2 + C\varepsilon^8 |\varphi_1|_8^8 y_\varepsilon + C\varepsilon^5 |\partial_z \varphi_1|_4^4 y_\varepsilon^3 + C\varepsilon^{\frac{3}{4}} y_\varepsilon z_\varepsilon,
\end{aligned} \tag{6.17}$$

and we have already seen that L_7 and L_8 satisfy exactly the same estimates than K_5 and K_6, and L_5 the same than K_3.

It remains to study the two terms L_1 and L_2. As far as the first one is concerned, we have to proceed in a really different way than what precedes, using fundamentally the new coercive terms in the energy estimate. More precisely, one has

$$|L_1| \le (\varepsilon |\partial_x R_\varepsilon|_2)(|\partial_z R_\varepsilon|_2) \le C z_\varepsilon.$$

Then, let us remark that we can choose this constant C to be as small as we want ($C \le 1/50$ for instance) by multiplying the original estimate (4.9) on $|R_\varepsilon|_2$ by a constant β large enough depending only on $|F''(\varphi_0)| = |F''(\omega)|$. Indeed, the term L_1 comes from the estimate on $|\partial_z R_\varepsilon|_2^2$, and the previous bound uses only the coercive terms coming from the estimate on $|R_\varepsilon|_2^2$.

All the estimates that we made previously do not change (the constants are just changed). Then, after this modification of the functional y_ε and z_ε, the estimate on L_1 reads

$$|L_1| \le \frac{1}{50} z_\varepsilon. \tag{6.18}$$

Finally, in a similar way, the second term is bounded by

$$|L_2| \le \varepsilon |u_0|_\infty (\varepsilon |\partial_x R_\varepsilon|_2) |\partial_z^2 R_\varepsilon|_2 \le C\varepsilon |u_0|_\infty z_\varepsilon. \tag{6.19}$$

To sum up the results, let us write the final estimate as follows:

$$\begin{aligned}
\frac{d}{dt} y_\varepsilon + \frac{1}{2} z_\varepsilon \le\ & (|\varphi_0|_\infty y_\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} y_\varepsilon + \varepsilon^{\frac{1}{2}} |\varphi_1^0|_2 + \varepsilon^{\frac{1}{4}} y_\varepsilon^{\frac{1}{2}} + C|\varphi_0|_\infty \varepsilon^{\frac{1}{2}} y_\varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{3}{4}} y_\varepsilon) z_\varepsilon \\
& + g_1(t) + g_2(t) y_\varepsilon^{\frac{1}{3}} + g_4(t) y_\varepsilon^3 + g_5(t) y_\varepsilon,
\end{aligned} \tag{6.20}$$

with

$$g_1(t) = \varepsilon^2 |\varphi_0|_\infty^2 |\varphi_1|_4^2 (\varepsilon^2 |\partial_x \varphi_1|_4^2 + |\partial_z \varphi_1|_4^2) + C\varepsilon^4 |\varphi_1|_6^6 + C\varepsilon^5 |\varphi_1|_8^4 (\varepsilon^2 |\partial_x \varphi_1|_4^2 + |\partial_z \varphi_1|_4^2),$$

$$g_2(t) = C\varepsilon^{\frac{7}{3}} |\partial_z^2 \varphi_1|_2^{\frac{4}{3}} |\partial_x \varphi_1|_4^{\frac{4}{3}} + C\varepsilon^{\frac{15}{3}} |\partial_x^2 \varphi_1|_2^{\frac{4}{3}} |\partial_x \varphi_1|_4^{\frac{4}{3}} + C\varepsilon |\partial_z^2 \varphi_1|_2^{\frac{4}{3}} |\partial_z \varphi_1|_4^{\frac{4}{3}} + C\varepsilon^{\frac{11}{3}} |\partial_x^2 \varphi_1|_2^{\frac{4}{3}} |\partial_z \varphi_1|_4^{\frac{4}{3}},$$

$$g_4(t) = C\varepsilon^9 |\partial_x \varphi_1|_4^4 + C\varepsilon^5 |\partial_z \varphi_1|_4^4,$$

$$g_5(t) = C\varepsilon^{\frac{1}{2}} (\varepsilon^2 |\partial_x^2 \varphi_1|_2 + |\partial_z^2 \varphi_1|_2)^2 + C\varepsilon^3 (\varepsilon |\partial_x \varphi_1|_4 + |\partial_z \varphi_1|_4)^4 + C\varepsilon^4 |\varphi_0|_\infty^4 |\varphi_1|_4^4 + C\varepsilon^8 |\varphi_1|_8^8.$$

Thanks to the uniform bounds in time on φ_1 (lemma 3.2), we can obtain some uniform estimates on the functions g_i in $L^1(\mathbb{R}^+)$.

Let us remark that, as φ_1 has zero average, we have the following Poincaré inequality:

$$|\varphi_1|_2 \leq C(|\partial_x \varphi_1|_2 + |\partial_z \varphi_1|_2),$$

which allows us to obtain, with the results of the second part of lemma 3.2, the bound

$$\int_0^{+\infty} \varepsilon^2 |\varphi_1|_2^2 dt \leq C |\varphi_1^0|_2^2,$$

that we will use systematically in what follows.

● **Estimates on g_1**

Thanks to the anisotropic Sobolev inequalities (3.11) and (3.13), one has

$$\begin{aligned}
\varepsilon^{\frac{13}{4}} \int_0^{+\infty} |\varphi_0|_\infty^2 |\varphi_1|_4^2 |\partial_x \varphi_1|_4^2 dt \leq C_0 \int_0^{+\infty} & |\varphi_1|_2^{\frac{3}{2}} (\varepsilon^2 |\varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{4}} (\varepsilon |\varphi_1|_2 + \varepsilon |\partial_z^2 \varphi_1|_2)^{\frac{1}{4}} \\
& \times |\varphi_1|_2^{\frac{1}{2}} (\varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^2 |\partial_x \varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} (\varepsilon |\partial_x \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2)^{\frac{1}{2}} dt \\
\leq C_0 |\varphi_1^0|_2^2 \int_0^{+\infty} & (\varepsilon^2 |\varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{4}} (\varepsilon |\varphi_1|_2 + \varepsilon |\partial_z^2 \varphi_1|_2)^{\frac{1}{4}} \\
& \times |\varphi_1|_2^{\frac{1}{2}} (\varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^2 |\partial_x \varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)^{\frac{1}{2}} (\varepsilon |\partial_x \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2)^{\frac{1}{2}} dt \\
\leq C_0 |\varphi_1^0|_2^4, &
\end{aligned}$$

and also

$$\varepsilon^{\frac{5}{4}} \int_0^{+\infty} |\varphi_0|_\infty^2 |\varphi_1|_4^2 |\partial_z \varphi_1|_4^2 dt \leq C_0 |\varphi_1^0|_2^4.$$

Similarly, one can show that

$$\varepsilon^3 \int_0^{+\infty} |\varphi_1|_6^6 dt \leq C_0 \int_0^{+\infty} |\varphi_1|_2^4 (\varepsilon^2 |\varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)(\varepsilon |\varphi_1|_2 + \varepsilon |\partial_z^2 \varphi_1|_2) dt \leq C_0 |\varphi_1^0|_2^6.$$

Finally, with the same kind of control that we have obtained in the study of the general case (section 5), it follows immediately that

$$\varepsilon^{\frac{15}{4}} \int_0^{+\infty} |\varphi_1|_8^4 |\partial_x \varphi_1|_4^2 dt \leq C_0 |\varphi_1^0|_2^4 (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2)^2,$$

and

$$\varepsilon^{\frac{7}{4}} \int_0^{+\infty} |\varphi_1|_8^4 |\partial_z \varphi_1|_4^2 dt \leq C_0 |\varphi_1^0|_2^4 (\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + |\varphi_1^0|_2)^2.$$

● **Estimates on g_2**

These are exactly the same as in the general case (except for the uniformity in time of course). We have

$$\int_0^{+\infty} g_2 dt \leq C_0 \varepsilon^{\frac{1}{3}} |\varphi_1^0|_2^2 (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2 + \varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + |\varphi_1^0|_2)^{\frac{2}{3}}.$$

• **Estimates on g_4**

Using the Sobolev inequality (3.11) with $p=4$, we have

$$\varepsilon^6 \int_0^{+\infty} |\partial_x \varphi_1|_4^4 dt \leq \int_0^{+\infty} (\varepsilon^3 |\partial_x \varphi_1|_2^2)(\varepsilon^2 |\partial_x \varphi_1|_2 + \varepsilon^2 |\partial_x^2 \varphi_1|_2)(\varepsilon |\partial_x \varphi_1|_2 + \varepsilon |\partial_x \partial_z \varphi_1|_2) dt \leq C_0 (\varepsilon^3 |\partial_x \varphi_1^0|_2^2) |\varphi_1^0|_2^2,$$

and

$$\varepsilon^2 \int_0^{+\infty} |\partial_x \varphi_1|_4^4 dt \leq C_0 (\varepsilon |\partial_z \varphi_1^0|_2^2 + |\varphi_1^0|_2^2) |\varphi_1^0|_2^2.$$

• **Estimates on g_5**

The first term is controlled immediately thanks to lemma 3.2 and the second one has just been treated in the study of g_4. The third one is bounded by

$$\varepsilon^2 \int_0^{+\infty} |\varphi_0|_\infty^2 |\varphi_1|_4^4 dt \leq C_0 \int_0^{+\infty} |\varphi_1|_2^2 (\varepsilon |\varphi_1|_2 + \varepsilon |\partial_x \varphi_1|_2)(\varepsilon |\varphi_1|_2 + \varepsilon |\partial_z \varphi_1|_2) dt \leq C_0 |\varphi_1^0|_2^4,$$

and the last one by

$$\begin{aligned}
\varepsilon^5 \int_0^{+\infty} |\varphi_1|_8^8 dt &\leq C \varepsilon^5 \int_0^{+\infty} |\varphi_1|_2^5 (|\varphi_1|_2 + |\partial_x^2 \varphi_1|_2)^{\frac{3}{2}} (|\varphi_1|_2 + |\partial_z^2 \varphi_1|_2)^{\frac{3}{2}} dt \\
&\leq C \int_0^{+\infty} |\varphi_1|_2^{\frac{9}{2}} (\varepsilon |\varphi_1|_2)^{\frac{1}{2}} \left(\varepsilon^2 |\varphi_1|_2^{\frac{1}{2}} (\varepsilon |\varphi_1|_2)^{\frac{1}{2}} + (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^{\frac{7}{2}} |\partial_x^3 \varphi_1|_2)^{\frac{1}{2}} \right)^{\frac{3}{2}} \\
&\qquad \times \left(|\varphi_1|_2^{\frac{1}{2}} (\varepsilon |\varphi_1|_2)^{\frac{1}{2}} + (\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1|_2)^{\frac{1}{2}} (\varepsilon^{\frac{1}{2}} |\partial_z^3 \varphi_1|_2)^{\frac{1}{2}} \right)^{\frac{3}{2}} dt \\
&\leq C_0 |\varphi_1^0|_2^5 (|\varphi_1^0|_2^3 + (\varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2)^3 + (\varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2)^3).
\end{aligned}$$

We have now collected all the inequalities we need to establish the proof of theorem 2.3.

Proof (of theorem 2.3):

As for the previous theorems, we remark that assumption (2.11) can written in the new variables under the form

$$|\varphi_1^0|_2 + \varepsilon^{\frac{1}{2}} |\partial_z \varphi_1^0|_2 + \varepsilon^{\frac{3}{2}} |\partial_x \varphi_1^0|_2 \leq K_0,$$

uniformly in ε. Then, the results obtained on the g_i previously read

$$\int_0^{+\infty} g_1 dt \leq \varepsilon^{\frac{3}{4}} K_0', \quad \int_0^{+\infty} g_2 dt \leq \varepsilon^{\frac{1}{3}} K_0',$$

$$\int_0^{+\infty} g_4 dt \leq \varepsilon^3 K_0', \quad \int_0^{+\infty} g_5 dt \leq \varepsilon^{\frac{1}{2}} K_0',$$

where K_0' is a constant depending only on K_0.

Let us now use the same kind of argument as in the previous proofs. We introduce

$$M_\varepsilon = 2(y_\varepsilon(0) + \varepsilon^{\frac{3}{4}} K_0' + \varepsilon^{\frac{1}{2}} K_0') e^{\varepsilon^3 K_0' + \varepsilon^{\frac{1}{2}} K_0' + K_0'}.$$

Assumption (2.12) and the fact that $R_\varepsilon(0) = 0$ show that

$$y_\varepsilon(0) \leq \varepsilon^{\frac{1}{2}} K_0,$$

which proves the existence of a constant M_0 depending only on φ_0, u_0 and K_0 such that, for any $\varepsilon > 0$, we have

$$M_\varepsilon \leq \varepsilon^{\frac{1}{2}} M_0.$$

Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, we have

$$M_\varepsilon \leq 1,$$

and

$$(|\varphi_0|_\infty M_\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} M_\varepsilon + \varepsilon^{\frac{1}{2}} |\varphi_1^0|_2 + \varepsilon^{\frac{1}{4}} M_\varepsilon^{\frac{1}{2}} + C|\varphi_0|_\infty \varepsilon^{\frac{1}{2}} M_\varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{3}{4}} M_\varepsilon) \leq \frac{1}{4}.$$

Let $T^* \in]0, +\infty]$ be the maximal time for which we have, for any $t \in [0, T^*]$,

$$y_\varepsilon(t) \leq M_\varepsilon.$$

Thanks to the previous properties of M_ε for $\varepsilon < \varepsilon_0$ and Young's inequality

$$\varepsilon^{\frac{1}{3}} y_\varepsilon^{\frac{1}{3}} \leq \varepsilon^{\frac{1}{2}} + y_\varepsilon,$$

the energy estimate can now be written on $[0, T^*]$ as follows:

$$\frac{d}{dt} y_\varepsilon \leq g_1(t) + \varepsilon^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2(t) \right) + \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2(t) + g_4(t) + g_5(t) \right) y_\varepsilon.$$

After integration in time, thanks to the bounds on the g_i previously obtained, one has for any $t \leq T^*$

$$\begin{aligned} y_\varepsilon(t) &\leq y_\varepsilon(0) + \int_0^{T^*} g_1 ds + \varepsilon^{\frac{1}{2}} \int_0^{T^*} \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2 \right) ds + \int_0^t \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2 + g_4 + g_5 \right) y_\varepsilon ds \\ &\leq y_\varepsilon(0) + \varepsilon^{\frac{3}{4}} K_0' + \varepsilon^{\frac{1}{2}} K_0' + \int_0^t \left(\frac{1}{\varepsilon^{\frac{1}{3}}} g_2 + g_4 + g_5 \right) y_\varepsilon ds \end{aligned}$$

so that with Gronwall's lemma, we obtain

$$y_\varepsilon(t) \leq (y_\varepsilon(0) + \varepsilon^{\frac{3}{4}} K_0' + \varepsilon^{\frac{1}{2}} K_0') e^{K_0' + \varepsilon^3 K_0' + \varepsilon^{\frac{1}{2}} K_0'} \leq \frac{1}{2} M_\varepsilon.$$

This proves that the maximal interval $[0, T^*]$ on which $y_\varepsilon(t) \leq M_\varepsilon$ is $\mathbb{R}^+$, that is to say $T^* = +\infty$. Hence, we have just proved that

$$\sup_{t \in \mathbb{R}^+} y_\varepsilon(t) \leq \varepsilon^{\frac{1}{2}} M_0,$$

which is exactly the claimed result if we come back to the original variables.

■

Acknowledgments: The authors want to thank professors A. Colin and D. Roux for many physical discussions of great interest. The authors want also to thank the referee for the care he takes to read this paper.

References

[1] N. Alikakos, P. Bates, and G. Fusco. Slow motion for the Cahn-Hilliard equation in one space dimension. *Journal of Differential Equation*, 90:pp 71–135, 1991.

[2] D.M. Anderson, G.B. McFadden, A.A. Wheeler. Diffuse-interfac methods in fluid mechanics. *Annu. Rev. Fluid. Mech.*, 30:pp 139-165, 1998.

[3] P. W. Bates and P. C. Fife. The dynamics of nucleation for the Cahn-Hilliard equation. *SIAM Journal of Applied Maths*, 53:pp 990–1008, 1993.

[4] T. Baumberger, F. Perrot, and D. Beyssens. *Phys. Rev. A*, 46:7636, 1992.

[5] D. Beyssens and M. Gladamassi. *J. Phys. Lett.*, 40:565, 1979.

[6] F. Boyer. Mathematical study of multiphase flow under shear through order parameter formulation. *Asymptotic Analysis*, 20:pp 175–212, 1999.

[7] F. Boyer. Nonhomogeneous Cahn-Hilliard fluids. *Annales de l'IHP : Analyse non linéaire*, 18(2), pp 225-259, 2001.

[8] F. Boyer. A theoretical and numerical model for the study of incompressible mixture flows. *Computers and Fluids*, 31(1), pp 41-68, 2002.

[9] J. W. Cahn, J. E. Hilliard, *Free energy of a Nonuniform System. III. Nucleation in a two-component incompressible fluid*, J. Chem. Phys. 31, pp. 688-699 (1959).

[10] J. Carr, M. Gurtin, M. Slemrod. Structured phase transitions on a finite interval. *Archive for Rational Mechanical Analysis*, 86:pp 317-351, 1984.

[11] R. Chella and J. Vinals. Mixing of a two-phase fluid by a cavity flow. *Physical Review E*, 53:p 3832, 1996.

[12] G. Cristobal Azkarate. *Etude de quelques changements structuraux dans les fluides complexes à l'équilibre et hors équilibre*. Ph.D thesis, CPMOH, Univ. Bordeaux 1, 2000.

[13] A. Debussche, L. Detorri. On the Cahn-Hilliard equation with logarithmic free-energy. *Nonlinear Analysis*, 10:pp 1491-1514, 1995.

[14] M. Doi. Dynamics of domain and textures. *Theoretical Challenges in th Dynamics of Complex Fluids*, pp 293-314, 1997.

[15] C.M. Elliott, H. Garcke On the Cahn-Hilliard equation with degenerate mobility. *SIAM Journal on Mathematical Analysis*, 27:pp 404-423, 1996.

[16] K. Matsuzaka, T. Koga, and T. Hashimoto. *Phys. Rev. Lett.*, 80:5441, 1998.

[17] D. Jacqmin. Calculation of two-phase Navier-Stokes flows using phase-field modelling. *Journal of Computational Physics*, 155:pp 96-127, 1999.

[18] A. Miranville. Some generalizations of the Cahn-Hilliard equation. *Asymptotic Analysis*, 22(3-4):pp 235-259, 2000.

[19] A. Miranville. Long-time behavior of some models of Cahn-Hilliard equations in a deformable continua. *Nonlinear Anal. Real World Appl.*, 2(3):pp 273-304 (2001).

[20] A. Novick-Cohen. On the viscous Cahn-Hilliard equation. *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford University Press, pp 329-342, 1988.

[21] A. Onuki. Phase transitions of fluids in shear flow. *J. Phys. : Condens. Matter*, 9:pp 6119-6157, 1997.

[22] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Science*. Springer-Verlag, 1997.

[23] R. Temam and M. Ziane. Navier-Stokes equations in three-dimensional thin domains with various boundary conditions. *Adv. Differential Equations*, 1(4):pp 499–546, 1996.

[24] J. Wei and M. Winter. Stationary solutions for the Cahn-Hilliard equation. *Ann. Inst. Henri Poincaré : Analyse non linéaire*, 15:pp 459–492, 1998.